

ORDERING LUSZTIG'S FAMILIES IN TYPE B_n

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ABSTRACT. Let W be a finite Coxeter group and L be a weight function on W in the sense of Lusztig. We have recently introduced a pre-order relation \preceq_L on the set of irreducible characters of W which extends Lusztig's definition of "families" and which, conjecturally, corresponds to the ordering given by Kazhdan–Lusztig cells. Here, we give an explicit description of \preceq_L for W of type B_n and any L . (All other cases are known from previous work.) This crucially relies on some new combinatorial constructions around Lusztig's "symbols". Combined with previous work, we deduce general compatibility results between \preceq_L and Lusztig's \mathbf{a} -function, valid for any W, L .

1. INTRODUCTION

Let W be a finite Coxeter group and $\text{Irr}(W)$ be the set of (complex) irreducible characters of W . Given a weight function L on W in the sense of Lusztig, one can attach to each $E \in \text{Irr}(W)$ a numerical invariant \mathbf{a}_E . These invariants are defined using the associated generic Iwahori–Hecke algebra; they are an essential ingredient in Lusztig's definition [20] of the "families" of $\text{Irr}(W)$. In [13], we have introduced a natural pre-order relation \preceq_L on $\text{Irr}(W)$ which extends the definition of families. Furthermore, we have shown that if we are in the "equal parameter case" (that is, L is constant on the generators of W) and W is the Weyl group of a connected reductive algebraic group G , then

- \preceq_L corresponds to the ordering given by Kazhdan–Lusztig cells;
- \preceq_L admits a geometric interpretation in terms of the Springer correspondence and the closure relation among special unipotent classes of G .

In particular, since the latter is known, this yields an explicit description of \preceq_L in the equal parameter case. The pre-orders associated with arbitrary L are not known to admit such a geometric interpretation, but in any case they are highly relevant in the study of cellular structures on Hecke algebras, following [11], [12].

Some further examples of \preceq_L have been discussed in [13] for cases where computations are possible (including Coxeter groups of type F_4 and of non-crystallographic type), but the essential case that remains to be considered – and this is what we will do in this paper – is when W is of type B_n and L is an arbitrary weight function on W . In this case, the set $\text{Irr}(W)$ is parametrised by pairs of partitions (λ, μ) such that $|\lambda| + |\mu| = n$ and there are infinitely many weight functions, indexed by two parameters $a, b \in \mathbb{Z}$. Thus, \preceq_L gives rise to pre-order relations on pairs of partitions, depending on the two parameters a, b . Our aim is to provide an explicit combinatorial description for these pre-orders.

A familiar example of a pre-order relation on pairs of partitions is given by the "dominance order" which already appeared in the work of Dipper–James–Murphy

[7]. We shall see that, in our setting, this corresponds precisely to the “asymptotic case” where b is large with respect to a (more precisely, $b > (n-1)a > 0$). At another extreme where $a = b$, our pre-order admits a geometric interpretation as mentioned above. The main results of this paper will deal with general choices of a, b . See also Bonnafé–Jacon [4] for a discussion of the combinatorics around the associated cellular structures in these cases.

In Section 2, we recall the precise definition of Lusztig’s families and the pre-order \preceq_L on $\text{Irr}(W)$. Following [17, §6.5], [14, §4], we work with a definition of the invariants \mathbf{a}_E which is independent of the theory of generic Iwahori–Hecke algebras.

In Section 3, we study certain pre-order relations on pairs of partitions which are defined in a purely combinatorial fashion. This is done in the framework provided by the combinatorics of Lusztig’s “symbols”. The most difficult result about these pre-orders is Proposition 3.12 which generalises a familiar property of the dominance order for partitions. As far as we are aware, this is a new result; the proof will be given in Sections 4 and 5.

In Section 6, we establish the relation between the Dipper–James–Murphy dominance order and our pre-order relations.

In Section 7, we prove the main results for W of type B_n , which show that the combinatorial constructions in Section 3 indeed describe the pre-order \preceq_L . This will be complemented in Section 8 by the discussion of examples and further interpretations of \preceq_L in type B_n . We conclude by establishing some general properties of \preceq_L in Section 9.

Combining these new results with the known ones from [13], we can draw some general conclusions about the pre-order relation \preceq_L , for any W and any weight function L as above. As Lusztig [20, Chap. 4] has shown, the \mathbf{a} -function is constant on the families of $\text{Irr}(W)$; this is one of the key properties of this function. In Section 9, we obtain the following refinement:

Let $E, E' \in \text{Irr}(W)$ be such that $E \preceq_L E'$. Then $\mathbf{a}_{E'} \leq \mathbf{a}_E$, with equality if and only if E, E' belong to the same Lusztig family.

We remark that, conjecturally, \preceq_L should coincide with the Kazhdan–Lusztig pre-order relation $\leq_{\mathcal{LR}}$, defined using the Kazhdan–Lusztig basis of the associated generic Iwahori–Hecke algebra; see Remark 2.6. It is part of Lusztig’s general conjectures in [21, §14.2] that the \mathbf{a} -function should satisfy a monotony property as above but with respect to $\leq_{\mathcal{LR}}$. Thus, it is our hope that the results in [13] and in this paper might provide a step towards a proof of Lusztig’s conjectures.

2. LUSZTIG’S \mathbf{a} -INVARIANTS AND FAMILIES

Let W be a finite Coxeter group, with generating set S and corresponding length function $l: W \rightarrow \mathbb{Z}_{\geq 0}$. Let $L: W \rightarrow \mathbb{Z}$ be a weight function, that is, we have $L(ww') = L(w) + L(w')$ whenever $w, w' \in W$ are such that $l(ww') = l(w) + l(w')$. (This is the setting of Lusztig [21].)

Throughout this paper, we shall assume that

$$\boxed{L(s) \geq 0 \quad \text{for all } s \in S.}$$

If, furthermore, there is some $a \in \mathbb{Z}$ such that $a > 0$ and $L(s) = a$ for all $s \in S$, then we say that we are in the *equal parameter case*.

Let $\text{Irr}(W)$ be the set of (complex) irreducible representations of W (up to isomorphism). Having fixed L as above, we shall define a function

$$\text{Irr}(W) \rightarrow \mathbb{Z}_{\geq 0}, \quad E \mapsto \mathbf{a}_E.$$

We need one further piece of notation. Let $T = \{ws w^{-1} \mid w \in W, s \in S\}$ be the set of all reflections in W . Let $S' \subseteq S$ be a set of representatives of the conjugacy classes of W which are contained in T . For $s \in S'$, let N_s be the cardinality of the conjugacy class of s ; thus, $|T| = \sum_{s \in S'} N_s$. Now let $E \in \text{Irr}(W)$ and $s \in S'$. Since s has order 2, it is clear that $\text{trace}(s, E) \in \mathbb{Z}$. Hence, by a well-known result in the character theory of finite groups, the quantity $N_s \text{trace}(s, E) / \dim E$ is an integer. Thus, we can define

$$\omega_L(E) := \sum_{s \in S'} \frac{N_s \text{trace}(s, E)}{\dim E} L(s) \in \mathbb{Z}.$$

(Note that this does not depend on the choice of the set of representatives $S' \subseteq S$.)

Definition 2.1. We define a function $\text{Irr}(W) \rightarrow \mathbb{Z}$, $E \mapsto \mathbf{a}_E$, inductively as follows. If $W = \{1\}$, then $\text{Irr}(W)$ only consists of the unit representation (denoted 1_W) and we set $\mathbf{a}_{1_W} := 0$. Now assume that $W \neq \{1\}$ and that the function $M \mapsto \mathbf{a}_M$ has already been defined for all proper parabolic subgroups of W . Then, for any $E \in \text{Irr}(W)$, we can define

$$\mathbf{a}'_E := \max\{\mathbf{a}_M \mid M \in \text{Irr}(W_J) \text{ where } J \subsetneq S \text{ and } M \uparrow E\}.$$

The notation $M \uparrow E$ is a shorthand for $E \mid \text{Ind}_{W_J}^W(M)$.

Finally, we set

$$\mathbf{a}_E := \begin{cases} \mathbf{a}'_E & \text{if } \mathbf{a}'_{E \otimes \varepsilon} - \mathbf{a}'_E \leq \omega_L(E), \\ \mathbf{a}'_{E \otimes \varepsilon} - \omega_L(E) & \text{otherwise,} \end{cases}$$

where ε denotes the sign representation.

Remark 2.2. One immediately checks that the function $E \mapsto \mathbf{a}_E$ satisfies the following conditions:

$$\mathbf{a}_E \geq \mathbf{a}'_E \geq 0 \quad \text{and} \quad \mathbf{a}_{E \otimes \varepsilon} - \mathbf{a}_E = \omega_L(E) \quad \text{for all } E \in \text{Irr}(W).$$

This also shows that $\mathbf{a}_E \geq \mathbf{a}_M$ if $M \uparrow E$ where $M \in \text{Irr}(W_J)$ and $J \subsetneq S$.

Example 2.3. (a) If $L(s) = 0$ for all $s \in S$, then $\mathbf{a}_E = 0$ for any $E \in \text{Irr}(W)$.

(b) The unit representation has \mathbf{a} -invariant 0 and the sign representation has \mathbf{a} -invariant $L(w_0)$ where $w_0 \in W$ is the longest element; we have $0 \leq \mathbf{a}_E \leq L(w_0)$ for all $E \in \text{Irr}(W)$. (See [16, §1.3] for details.)

We just remark that Lusztig [19], [21] originally defined “ \mathbf{a} -invariants” \mathbf{a}_E using the “generic degrees” of the generic Iwahori–Hecke algebra associated with W, L (see Remark 9.4). The fact that this is equivalent to Definition 2.1 is shown in [14, Remark 4.3]. The above definition will be sufficient for the purposes of this paper.

It will be convenient to introduce the following notation. Let $J \subseteq S$, $M \in \text{Irr}(W_J)$ and $E \in \text{Irr}(W)$. Then we write $M \rightsquigarrow_L E$ if $M \uparrow E$ and $\mathbf{a}_M = \mathbf{a}_E$.

Definition 2.4 (Lusztig [20, 4.2]). The partition of $\text{Irr}(W)$ into “families” is defined as follows. When $W = \{1\}$, there is only one family; it consists of the unit representation of W . Assume now that $W \neq \{1\}$ and that families have already been defined for all proper parabolic subgroups of W . Then $E, E' \in \text{Irr}(W)$ are said to

be in the same family for $\text{Irr}(W)$ if there exists a sequence $E = E_0, E_1, \dots, E_m = E'$ in $\text{Irr}(W)$ such that, for each $i \in \{1, 2, \dots, m\}$, the following condition is satisfied. There exists a subset $I_i \subsetneq S$ and $M_i, M'_i \in \text{Irr}(W_{I_i})$, where M_i, M'_i belong to the same family of $\text{Irr}(W_{I_i})$, such that either

$$M_i \rightsquigarrow_L E_{i-1} \quad \text{and} \quad M'_i \rightsquigarrow_L E_i$$

or

$$M_i \rightsquigarrow_L E_{i-1} \otimes \varepsilon \quad \text{and} \quad M'_i \rightsquigarrow_L E_i \otimes \varepsilon.$$

Note that it is clear from this definition that, if $\mathcal{F} \subseteq \text{Irr}(W)$ is a family, then $\mathcal{F} \otimes \varepsilon := \{E \otimes \varepsilon \mid E \in \mathcal{F}\}$ is a family, too.

Definition 2.5. Following [13], we define a relation \preceq_L on $\text{Irr}(W)$ inductively as follows. If $W = \{1\}$, then $\text{Irr}(W)$ only consists of the unit representation and this is related to itself. Now assume that $W \neq \{1\}$ and that \preceq_L has already been defined for all proper parabolic subgroups of W . Let $E, E' \in \text{Irr}(W)$. Then we write $E \preceq_L E'$ if there is a sequence $E = E_0, E_1, \dots, E_m = E'$ in $\text{Irr}(W)$ such that, for each $i \in \{1, 2, \dots, m\}$, the following condition is satisfied. There exists a subset $I_i \subsetneq S$ and $M_i, M'_i \in \text{Irr}(W_{I_i})$, where $M_i \preceq_L M'_i$ within $\text{Irr}(W_{I_i})$, such that either

$$M_i \uparrow E_{i-1} \quad \text{and} \quad M'_i \rightsquigarrow_L E_i$$

or

$$M_i \uparrow E_i \otimes \varepsilon \quad \text{and} \quad M'_i \rightsquigarrow_L E_{i-1} \otimes \varepsilon.$$

It is already remarked in [13] that, if E, E' belong to the same family, then $E \preceq_L E'$ and $E' \preceq_L E$. In Corollary 9.2 we will see that the converse also holds. Note that this is not clear from the definitions.

Remark 2.6. Given the weight function L , one can define pre-order relations $\leq_{\mathcal{L}}$, $\leq_{\mathcal{R}}$, $\leq_{\mathcal{LR}}$ on W , using the Kazhdan–Lusztig basis of the associated Iwahori–Hecke algebra. The corresponding equivalence classes are called the left, right and two-sided cells of W ; see Lusztig [21]. As explained in [13, §2], the two-sided relation $\leq_{\mathcal{LR}}$ on W induces a pre-order relation on $\text{Irr}(W)$ which we denote by the same symbol. It is shown in [13, Prop. 3.4] that we have the following implication, where $E, E' \in \text{Irr}(W)$:

$$E \preceq_L E' \quad \Rightarrow \quad E \leq_{\mathcal{LR}} E'.$$

Conjecturally, the reverse implication should also hold. In the equal parameter case, this is proved in [13, Theorem 4.10]. As far as unequal parameters are concerned, it is known to be true for any L and W of type F_4 or $I_2(m)$; see [13, §3]. In Example 8.1, we will see that this also holds for an infinite collection of weight functions in type B_n . A general proof of this conjecture, for any weight function L , would be a major breakthrough in this theory.

Example 2.7. Assume that $L(s) = 0$ for all $s \in S$. By Example 2.3 we have $\mathbf{a}_E = 0$ for all $E \in \text{Irr}(W)$. This implies that $E \preceq_L E'$ for all $E, E' \in \text{Irr}(W)$; in particular, the whole set $\text{Irr}(W)$ is a family.

Before we give further (and non-trivial) examples, we discuss two reduction statements which will be helpful in the determination of the relation \preceq_L .

Remark 2.8. Let $W = W_1 \times \cdots \times W_k$ be the decomposition of W into irreducible components. Correspondingly, we have

$$\text{Irr}(W) = \{E_1 \boxtimes \cdots \boxtimes E_k \mid E_i \in \text{Irr}(W_i) \text{ for } 1 \leq i \leq k\}.$$

Now the function $E \mapsto \mathbf{a}_E$ is easily seen to be compatible with the above decomposition, that is, we have $\mathbf{a}_E = \mathbf{a}_{E_1} + \cdots + \mathbf{a}_{E_k}$ where \mathbf{a}_{E_i} is defined with respect to the restriction of L to W_i . Furthermore, the induction of representations from parabolic subgroups is compatible with the above decomposition. Consequently, the following hold, where $E_i, E'_i \in \text{Irr}(W_i)$ for $1 \leq i \leq k$.

- (a) $E_1 \boxtimes \cdots \boxtimes E_k$ and $E'_1 \boxtimes \cdots \boxtimes E'_k$ belong to the same family of $\text{Irr}(W)$ if and only if E_i and E'_i belong to the same family of $\text{Irr}(W_i)$ for $1 \leq i \leq k$.
- (b) $E_1 \boxtimes \cdots \boxtimes E_k \preceq_L E'_1 \boxtimes \cdots \boxtimes E'_k$ within $\text{Irr}(W)$ if and only if $E_i \preceq_L E'_i$ within $\text{Irr}(W_i)$ for $1 \leq i \leq k$.

Hence, it is sufficient to determine the pre-order relation \preceq_L in the case where (W, S) is irreducible.

Remark 2.9. Let $E \in \text{Irr}(W)$ and assume that there exists a proper subset $I \subsetneq S$ and $M \in \text{Irr}(W_I)$ such that $M \uparrow E$. Now let $I_1 \subsetneq S$ such that $I \subseteq I_1$. Then, by the transitivity of induction, it is clear that there exists some $M_1 \in \text{Irr}(W_{I_1})$ such that

- (a) $M \uparrow M_1$ and $M_1 \uparrow E$.

Now assume that $M \rightsquigarrow_L E$. Then, by Remark 2.2 and the transitivity of induction, there exists some $M_1 \in \text{Irr}(W_{I_1})$ such that

- (b) $M \rightsquigarrow_L M_1$ and $M_1 \rightsquigarrow_L E$.

These relations immediately imply that, in the formulation of Definition 2.5, we may assume without loss of generality that each W_{I_i} is a maximal parabolic subgroup of W . (A similar statement concerning families already appeared in [20, 4.2].)

Remark 2.10. Assume that (W, S) is irreducible and that $L(s) > 0$ for all $s \in S$. The invariants \mathbf{a}_E and the families are explicitly known in all cases by the work of Lusztig; see [20, Chap. 4], [17, §6.5] (for the equal parameter case) and [21, Chap. 23] (for the remaining cases). The article [13] explicitly describes the pre-order relation \preceq_L on $\text{Irr}(W)$, except for the case where W is of type B_n and we are not in the equal parameter case.

Example 2.11. Let $n \geq 1$ and (W, S) be of type A_{n-1} . Then W can be identified with the symmetric group \mathfrak{S}_n and $\text{Irr}(W)$ is parametrised by the partitions of n . Thus, we can write

$$\text{Irr}(W) = \{E^\lambda \mid \lambda \text{ is a partition of } n\}.$$

For example, the unit representation is labelled by (n) and the sign representation is labelled by (1^n) . Assume now that L is not identically zero. All generators in S are conjugate in W and so there is some $a \in \mathbb{Z}_{>0}$ such that $L(s) = a$ for all $s \in S$.

Let λ be a partition of n . Writing the parts of λ as $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$ where $N \geq 1$, we set

$$n(\lambda) := \sum_{1 \leq i \leq N} (i-1)\lambda_i.$$

Then we have $\mathbf{a}_{E^\lambda} = n(\lambda)a$; see, for example, [17, 6.5.8, 9.4.5]. By Lusztig [20, 4.4], each family of $\text{Irr}(W)$ consists of a single representation. Now recall that the

dominance order \leq on partitions is defined as follows. Let λ, μ be two partitions of n . By adding zeroes if necessary, we can write

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0) \quad \text{and} \quad \mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_N \geq 0)$$

for some $N \geq 1$. Then we have

$$\lambda \leq \mu \stackrel{\text{def}}{\iff} \sum_{1 \leq i \leq d} \lambda_i \leq \sum_{1 \leq i \leq d} \mu_i \quad \text{for all } d \in \{1, \dots, N\}.$$

With this notation, the following three conditions are equivalent:

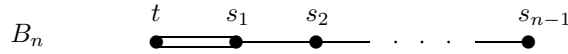
- (a) $E^\lambda \preceq_L E^\mu$.
- (b) $\lambda \leq \mu$.
- (c) There exists a subset $I \subseteq S$ such that $\varepsilon_I \uparrow E^\lambda$ and $\varepsilon_I \rightsquigarrow_L E^\mu$.

Proof. “(a) \Rightarrow (b)” By Remark 2.6, the assumption implies that $E^\lambda \leq_{\mathcal{LR}} E^\mu$. Then (b) holds by [10, Theorem 5.1]; see also [16, §2.8]. (Note that no geometric interpretation is required here.)

“(b) \Rightarrow (c)” Let $W_I \subseteq W$ be the parabolic subgroup which corresponds, under the identification $W \cong \mathfrak{S}_n$, to the Young subgroup $\mathfrak{S}_{\bar{\mu}}$ where $\bar{\mu}$ denotes the conjugate partition. Then it is well-known that $\varepsilon_I \rightsquigarrow_L E^\mu$; see, for example, [17, Prop. 5.4.7]. So it remains to show that $\varepsilon_I \uparrow E^\lambda$. Now, since $\lambda \leq \mu$, we also have $\bar{\mu} \leq \bar{\lambda}$; see [22, I.1.11]. Consequently, the Kostka number $\kappa_{\bar{\lambda}, \bar{\mu}}$ is non-zero; see [22, I.6.4]. This implies that $1_I \uparrow E^{\bar{\lambda}}$ where 1_I stands for the unit representation; see the remark following [22, I.7.8]. Now note that, by [17, Cor. 5.4.9], we have $E^\lambda = E^{\bar{\lambda}} \otimes \varepsilon$; furthermore, $\text{Ind}_I^S(\varepsilon_I) = \text{Ind}_I^S(1_I) \otimes \varepsilon$. Consequently, we also have $\varepsilon_I \uparrow E^\lambda$, as required.

“(c) \Rightarrow (a)” This is clear by the definition of \preceq_L . \square

Example 2.12. Let $n \geq 1$ and $W = W_n$ be a Coxeter group of type B_n , with generators and diagram given by



Then $\text{Irr}(W_n)$ is naturally parametrised by bipartitions of n , that is, pairs of partitions (λ, μ) such that $|\lambda| + |\mu| = n$; we shall write

$$\text{Irr}(W_n) = \{E^{(\lambda, \mu)} \mid (\lambda, \mu) \text{ is a bipartition of } n\}.$$

For example, the unit and the sign representation are labelled by $((n), \emptyset)$ and $(\emptyset, (1^n))$, respectively; see [17, §5.5]. A weight function $L: W_n \rightarrow \mathbb{Z}$ is specified by the two parameters

$$b := L(t) \geq 0 \quad \text{and} \quad a := L(s_i) \geq 0 \quad \text{for } 1 \leq i \leq n-1.$$

Thus, the relation \preceq_L will depend on a, b ; simple examples show that \preceq_L is really different for different values of a, b .

In the next section, we will begin with the study of this case by considering certain pre-order relations $\preceq_{a,b}$ on bipartitions. These will turn out to be the key for describing the relation \preceq_L on $\text{Irr}(W_n)$; see Theorem 7.11.

3. ORDERING BIPARTITIONS

The aim of this section is to define suitable generalisations of the dominance order for partitions to the setting of bipartitions of n . The framework for doing this is provided by the combinatorics developed by Lusztig in [21, Chap. 22].

Let us fix some notation. As in Example 2.12, we shall fix two integers $a, b \in \mathbb{Z}$ such that $a > 0$ and $b \geq 0$. (The case where $a = 0$ will be treated separately; see Example 7.13.) Division with remainder defines then two integers $r, b' \geq 0$ by

$$b = ra + b' \quad \text{and} \quad 0 \leq b' < a.$$

Following Lusztig [21, 22.6], given any integer $N \geq 0$, we define $\mathcal{M}_{a,b;n}^N$ to be the set of all multisets $Z = \{z_1, z_2, \dots, z_{2N+r}\}$ such that the following hold:

(M1) The entries of Z are elements of $\mathbb{Z}_{\geq 0}$ and we have

$$\sum_{1 \leq i \leq 2N+r} z_i = na + N^2a + N(b-a) + \binom{r}{2}a + rb'.$$

(M2) If $b' = 0$, there are at least $N+r$ distinct entries in Z , no entry is repeated more than twice, and all entries of Z are contained in $\mathbb{Z}a$.

(M3) If $b' > 0$, then all entries of Z are distinct; furthermore, N entries of Z are contained in $\mathbb{Z}a$ and $N+r$ entries are contained in $b' + \mathbb{Z}a$.

There is a “shift” operation $\mathcal{M}_{a,b;n}^N \rightarrow \mathcal{M}_{a,b;n}^{N+1}$ given by

$$\{z_1, z_2, \dots, z_{2N+r}\} \mapsto \{0, b', z_1 + a, z_2 + a, \dots, z_{2N+r} + a\}.$$

Given two multisets $Z \in \mathcal{M}_{a,b;n}^N$ and $Z' \in \mathcal{M}_{a,b;n}^{N'}$ (where $N, N' \geq 0$ are integers), we write $Z \sim Z'$ if one of Z, Z' can be obtained from the other by a finite sequence of shift operations. This defines an equivalence relation on the union of all multisets $\bigcup_{N \geq 0} \mathcal{M}_{a,b;n}^N$. These constructions are related to bipartitions of n , in the following way.

Definition 3.1 (Lusztig). Let (λ, μ) be a bipartition of n . Choosing a sufficiently large integer $N \geq 0$ and adding zeroes if necessary, we can write

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{N+r} \geq 0) \quad \text{and} \quad \mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_N \geq 0).$$

Then we define $Z_{a,b}^N(\lambda, \mu)$ to be the multiset formed by the $2N+r$ entries

$$\begin{aligned} (\lambda_i + N + r - i)a + b' & \quad (1 \leq i \leq N+r), \\ (\mu_j + N - j)a & \quad (1 \leq j \leq N). \end{aligned}$$

As pointed out in [21, 22.10], we have $Z_{a,b}^N(\lambda, \mu) \in \mathcal{M}_{a,b;n}^N$. Furthermore, if we choose another N' , then $Z_{a,b}^N(\lambda, \mu) \sim Z_{a,b}^{N'}(\lambda, \mu)$.

The above array of $2N+r$ integers, arranged in two rows, is an example of Lusztig's “symbols”, which originally appeared in the classification of unipotent representations of finite classical groups; see [20, Chap. 4].

Example 3.2. Let us consider the bipartition $(4311, 32)$ of 14.

First assume that $a = b = 1$. Then $r = 1$ and $b' = 0$. We can take $N = 3$ and write our bipartition as $(4311, 320)$. The corresponding multiset is

$$Z_{1,1}^3(4311, 32) = \{7, 5, 5, 3, 2, 1, 0\}.$$

If we take $N = 5$, then we write our bipartition as $(431100, 32000)$ and the corresponding multiset is

$$Z_{1,1}^5(4311, 32) = \{9, 7, 7, 5, 4, 3, 2, 1, 1, 0, 0\}.$$

This multiset is obtained from the previous one by performing two shifts.

Now assume that $a = 2$, $b = 1$. Then $r = 0$ and $b' = 1$. We can take $N = 4$ and write our bipartition as $(4311, 3200)$. The corresponding multiset is

$$Z_{2,1}^4(4311, 32) = \{15, 12, 11, 8, 5, 3, 2, 0\}.$$

Remark 3.3. Let γ be the right hand side of the formula in (M1). If we arrange the entries of a multiset $Z \in \mathcal{M}_{a,b;n}^N$ in decreasing order, we obtain a partition of γ . Thus, all multisets in $\mathcal{M}_{a,b;n}^N$ can be regarded as partitions of γ . Consequently, the set $\mathcal{M}_{a,b;n}^N$ is partially ordered by the dominance order and it makes sense to write $Z \leq Z'$ for $Z, Z' \in \mathcal{M}_{a,b;n}^N$.

One easily checks that if $Z_1, Z_2 \in \mathcal{M}_{a,b;n}^N$ and $Z'_1, Z'_2 \in \mathcal{M}_{a,b;n}^{N'}$ (where $N, N' \geq 0$ are integers), then the following implication holds:

$$Z_1 \sim Z'_1, \quad Z'_2 \sim Z_2 \quad \text{and} \quad Z_1 \leq Z_2 \quad \Rightarrow \quad Z'_1 \leq Z'_2.$$

Thus, \leq can be regarded as a partial order on the equivalence classes of multisets as above, modulo the shift operation.

We are now ready to define the desired generalisations of the dominance order.

Definition 3.4. We define a pre-order relation $\preceq_{a,b}$ on the set of bipartitions of n , as follows. Let (λ, μ) and (λ', μ') be bipartitions of n . Choose a sufficiently large integer $N \geq 0$ and consider the multisets $Z_{a,b}^N(\lambda, \mu)$, $Z_{a,b}^N(\lambda', \mu')$ in Definition 3.1. Then

$$(\lambda, \mu) \preceq_{a,b} (\lambda', \mu') \quad \stackrel{\text{def}}{\Leftrightarrow} \quad Z_{a,b}^N(\lambda, \mu) \leq Z_{a,b}^N(\lambda', \mu').$$

We say that the bipartitions (λ, μ) and (λ', μ') belong to the same “combinatorial family” if $Z_{a,b}^N(\lambda, \mu) = Z_{a,b}^N(\lambda', \mu')$. Then $\preceq_{a,b}$ induces a partial order on the set of combinatorial families which we denote by the same symbol. Note that these definitions do not depend on the choice of N .

(In a somewhat different context, similar pre-orders on pairs of partitions are considered in [16, §5.5] and also by Chlouveraki et al. [6, §5].)

Remark 3.5. Assume that $b' > 0$. Let (λ, μ) be a bipartition of n and $Z_{a,b}^N(\lambda, \mu)$ be the corresponding multiset (for some $N \geq 0$). Then the expressions in Definition 3.1 show that (λ, μ) is uniquely determined by $Z_{a,b}^N(\lambda, \mu)$. Consequently, all combinatorial families are singleton sets in this case, and $\preceq_{a,b}$ is a partial order.

On the other hand, if $b' = 0$, then simple examples show that, in general, the combinatorial families will contain more than one element.

Also note that if $b' = 0$, then the following equivalence holds

$$(\lambda, \mu) \preceq_{a,b} (\lambda', \mu') \quad \Leftrightarrow \quad (\lambda, \mu) \preceq_{1,r} (\lambda', \mu')$$

for all bipartitions (λ, μ) and (λ', μ') of n .

Example 3.6. Assume that $a = 1$ and $b = n - 1$; we call this the “sub-asymptotic” case. We have $r = n - 1$ and $b' = 0$. Let us describe the combinatorial families in this case. Let $\mathcal{F}_0 := \{(1^k, l) \mid k, l \geq 0 \text{ and } k + l = n\}$. We claim that:

- (i) \mathcal{F}_0 is a combinatorial family;
- (ii) all the other combinatorial families are singleton sets.

Indeed, let $(\lambda, \mu) = (1^k, (l))$ where $n = k + l$. Let $N := n$. The corresponding multiset $Z_{a,b}^n(\lambda, \mu)$ contains the entries

$$\begin{aligned}\lambda_i + N + r - i &= \lambda_i + 2n - 1 - i & (1 \leq i \leq 2n - 1), \\ \mu_j + N - j &= \mu_j + n - j & (1 \leq j \leq n).\end{aligned}$$

The first row of the above array yields the entries $\{0, 1, 2, \dots, 2n - 1\} \setminus \{2n - k - 1\}$; the second row yields the entries $\{0, 1, 2, \dots, n - 2\} \cup \{l + n - 1\}$. Thus, since $l + n - 1 = 2n - k - 1$, we obtain

$$\begin{aligned}Z_{a,b}^n(\lambda, \mu) &= \{0, 1, 2, \dots, 2n - 1\} \cup \{0, 1, 2, \dots, n - 2\} \\ &= \{0, 0, 1, 1, 2, 2, \dots, n - 2, n - 2, n - 1, n, n + 1, \dots, 2n - 1\}.\end{aligned}$$

Hence, all bipartitions of the form $((1^k), (l))$ (where $n = k + l$) belong to the same combinatorial family. Conversely, let (λ, μ) be a bipartition which is not of this form. Suppose first that $\lambda_1 > 1$. Then $\lambda_1 + N + r - 1 > 2n - 1$ and so (λ, μ) is not in the combinatorial family of \mathcal{F}_0 . Similarly, if μ has at least two (non-zero) parts, then the sequence $\{\mu_j + N - j \mid 1 \leq j \leq N\}$ will not contain all of the numbers $0, 1, 2, \dots, n - 2$ and, hence, these numbers will not all be repeated twice in $Z_{a,b}^n(\lambda, \mu)$. So, again, (λ, μ) is not in the combinatorial family of \mathcal{F}_0 . Thus, (i) is proved. The proof of (ii) is a similar combinatorial exercise; the precise argument is along the lines of the discussion in Section 6. We omit further details.

Example 3.7. Assume that $a = 1$ and $b \in \{0, 1\}$. Thus, we are either in the equal parameter case or in the case which is relevant to groups of type D_n (see Example 8.6). We have $b' = 0$ and $r \in \{0, 1\}$.

Let (λ, μ) be a bipartition of n and write

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{N+r} \geq 0), \quad \mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_N \geq 0)$$

for some $N \geq 0$. As in [20, Chap. 4], we say that (λ, μ) is “special” (or “ (a, b) -special” if a, b are not clear from the context), if the following conditions hold:

$$\begin{aligned}\lambda_i + 1 &\geq \mu_i \geq \lambda_{i+1} & (1 \leq i \leq N) & \quad \text{if } a = b = 1, \\ \lambda_N &\geq \mu_N \text{ and } \lambda_i \geq \mu_i \geq \lambda_{i+1} - 1 & (1 \leq i \leq N - 1) & \quad \text{if } a = 1, b = 0.\end{aligned}$$

Then every combinatorial family contains a unique special bipartition.

Next recall from Example 2.11 the definition of the numerical invariant $n(\lambda)$ associated with a partition λ of n . This invariant has the following compatibility property with the dominance order: Let λ and μ be partitions of n such that $\lambda \leq \mu$. Then $n(\mu) \leq n(\lambda)$, with equality only for $\lambda = \mu$; see [17, Exc. 5.6]. We will now see that the \mathbf{a} -invariants attached to bipartitions by Lusztig [21, §22] satisfy a similar compatibility property.

Definition 3.8. Let $N \geq 0$ and $Z \in \mathcal{M}_{a,b;n}^N$. Let Z^0 be the unique multiset in $\mathcal{M}_{a,b;0}^N$; its entries are

$$0, a, 2a, \dots, (N - 1)a, b', a + b', 2a + b', \dots, (N + r - 1)a + b'.$$

(If $N = 0$, then these entries are $ia + b'$ for $0 \leq i \leq r - 1$.) Then we define

$$\mathbf{a}_{a,b}(Z) := \sum_{1 \leq i \leq 2N+r} (i - 1)z_i - \sum_{1 \leq i \leq 2N+r} (i - 1)z_i^0,$$

where $z_1, z_2, \dots, z_{2N+r}$ are the entries of Z and $z_1^0, z_2^0, \dots, z_{2N+r}^0$ are the entries of Z^0 , both arranged in decreasing order.

Now let (λ, μ) be a bipartition on n . Choose N sufficiently large and consider the multiset $Z_{a,b}^N(\lambda, \mu)$ in Definition 3.1. Then we define

$$\mathbf{a}_{a,b}(\lambda, \mu) := \mathbf{a}_{a,b}(Z_{a,b}^N(\lambda, \mu)).$$

Remark 3.9. (i) Note that the above formula coincides with that given by Lusztig [21, Prop. 22.14]; in particular, $\mathbf{a}_{a,b}(\lambda, \mu)$ does not depend on the choice of N .

(ii) Let $N \geq 0$ and $Z, Z' \in \mathcal{M}_{a,b;n}^N$ be such that $Z \trianglelefteq Z'$. Then $\mathbf{a}_{a,b}(Z') \leq \mathbf{a}_{a,b}(Z)$, with equality only if $Z = Z'$. (This follows by an argument entirely analogous to that for partitions and the invariants $n(\lambda)$; see [17, Exc. 5.6].)

(iii) Consequently, if (λ, μ) and (λ', μ') are bipartitions of n such that $(\lambda, \mu) \preceq_{a,b} (\lambda', \mu')$, then $\mathbf{a}_{a,b}(\lambda', \mu') \leq \mathbf{a}_{a,b}(\lambda, \mu)$, with equality only if (λ, μ) and (λ', μ') belong to the same combinatorial family.

Finally, we come to the most subtle property of the pre-order relation $\preceq_{a,b}$. Recall that, if λ and μ are partitions of n such that $\lambda \trianglelefteq \mu$, then we also have $\overline{\mu} \trianglelefteq \overline{\lambda}$; see [22, I.1.11]. Here, for any partition ν , we denote by $\overline{\nu}$ the conjugate partition. Our task is to generalise this to bipartitions. Following Lusztig [21, 22.8], we can define a conjugation operation on multisets as follows. Let $Z = \{z_1, z_2, \dots, z_{2N+r}\} \in \mathcal{M}_{a,b;n}^N$. Let $t \geq 0$ be an integer which is large enough such that the multiset

$$\{ta + b' - z_1, ta + b' - z_2, \dots, ta + b' - z_{2N+r}\}$$

is contained in the multiset

$$\{0, a, 2a, \dots, ta, b', a + b', 2a + b', \dots, ta + b'\}.$$

Then we define \overline{Z} to be the complement of the first of the above two multisets in the second. As pointed out in [21, 22.8], we have

$$\overline{Z} \in \mathcal{M}_{a,b;n}^{t+1-N-r}.$$

(Note that (i) $\overline{\overline{Z}} \sim Z$ and (ii) $\overline{Z}_{(t)} \sim \overline{Z}_{(t')}$, where $\overline{Z}_{(t)}$ and $\overline{Z}_{(t')}$ denote the conjugates of Z calculated for the integers t and t' respectively.)

Also note the following interpretation in terms of bipartitions.

Lemma 3.10. *Let (λ, μ) be a bipartition of n . Then $\overline{Z_{a,b}^N(\lambda, \mu)} \sim Z_{a,b}^N(\overline{\mu}, \overline{\lambda})$.*

Proof. This follows from [21, 22.18] and [17, 5.5.6]. \square

We can now state the following fundamental compatibility property.

Lemma 3.11. *In the above setting, let $Z, Z' \in \mathcal{M}_{a,b;n}^N$ be such that $Z \trianglelefteq Z'$. Let t be such that both \overline{Z} and \overline{Z}' are defined and in $\mathcal{M}_{a,b;n}^{t+1-N-r}$. Then we have $\overline{Z}' \trianglelefteq \overline{Z}$.*

Proof. For the proof we distinguish the two cases where either $b' = 0$ or $b' > 0$. The details of the argument will be given in Section 4. \square

As an immediate consequence we obtain the following key property of the relation $\preceq_{a,b}$. This is one of the crucial ingredients in the proof of Theorem 7.11.

Proposition 3.12. *Let (λ, μ) and (λ', μ') be bipartitions of n . Then*

$$(\lambda, \mu) \preceq_{a,b} (\lambda', \mu') \quad \Leftrightarrow \quad (\overline{\mu'}, \overline{\lambda'}) \preceq_{a,b} (\overline{\mu}, \overline{\lambda}).$$

Proof. This is clear by the definition of $\preceq_{a,b}$ and Lemmas 3.10 and 3.11. \square

4. PROOF OF LEMMA 3.11: THE CASE WHERE $b' = 0$

Throughout this section we assume that $b' = 0$. By Remark 3.5, we have

$$(\lambda, \mu) \preceq_{a,b} (\lambda', \mu') \Leftrightarrow (\lambda, \mu) \preceq_{1,r} (\lambda', \mu')$$

for all bipartitions (λ, μ) and (λ', μ') of n . Thus, in order to prove Lemma 3.11, we can assume without loss of generality that $a = 1$ and $b = r \geq 0$. Then $\mathcal{M}_{1,r;n}^N$ consists of multisets whose entries are non-negative integers such that the conditions (M1) and (M2) in Section 3 hold.

It will now be convenient to work with a slightly larger class of multisets. We define $\widetilde{\mathcal{M}}_{1,r;n}^N$ to be the set of all multisets whose entries are non-negative integers satisfying the condition (M1) and such that, instead of (M2), the weaker condition that no entry is repeated more than twice holds (but we do not make an assumption on the number of distinct entries). Thus, $\mathcal{M}_{a,b;n}^N \subseteq \widetilde{\mathcal{M}}_{a,b;n}^N$. We shall prove the statement in Lemma 3.11 for this larger class of multisets. The advantage of using $\widetilde{\mathcal{M}}_{a,b;n}^N$ instead of $\mathcal{M}_{a,b;n}^N$ is that we have the following simple description of adjacent multisets in $\widetilde{\mathcal{M}}_{a,b;n}^N$ with respect to \preceq .

Lemma 4.1. *Let $Z, Z' \in \widetilde{\mathcal{M}}_{a,b;n}^N$ be such that $Z \preceq Z'$ and $Z \neq Z'$. Assume that Z, Z' are adjacent with respect to \preceq (that is, if $Z \preceq Z'' \preceq Z'$ for some $Z'' \in \widetilde{\mathcal{M}}_{a,b;n}^N$, then $Z = Z''$ or $Z'' = Z'$). Write $Z = \{z_1 \geq z_2 \geq \dots \geq z_{2N+r}\}$ and $Z' = \{z'_1 \geq z'_2 \geq \dots \geq z'_{2N+r}\}$. Then there exist indices $1 \leq k < l \leq 2N+r$ such that $z'_k = z_k + 1$, $z'_l = z_l - 1$ and $z'_i = z_i$ for all $i \neq k, l$.*

Proof. This is similar to the proof of the analogous result for general partitions in [22, I.1.16].

Let $k \geq 1$ be such that $z_i = z'_i$ for $1 \leq i \leq k-1$ and $z_k < z'_k$. Note that, if $k \geq 2$, then this implies $z_k < z'_k \leq z'_{k-1} = z_{k-1}$. Furthermore, we have $z_1 + \dots + z_k < z'_1 + \dots + z'_k$. Let $l > k$ be minimal such that

$$z_1 + \dots + z_l = z'_1 + \dots + z'_l.$$

Then we have $z_1 + \dots + z_{l-1} < z'_1 + \dots + z'_{l-1}$ and so $z_l > z'_l$. Since $z_1 + \dots + z_{l+1} \leq z'_1 + \dots + z'_{l+1}$, we also have $z_{l+1} \leq z'_{l+1}$ and so

$$z_l > z'_l \geq z'_{l+1} \geq z_{l+1}.$$

We define a multiset $Z'' = \{z''_1, \dots, z''_{2N+r}\}$ by $z''_k := z_k + 1$, $z''_l := z_l - 1$ and $z''_i := z_i$ for all $i \neq k, l$. Note that $z''_1 \geq z''_2 \geq \dots \geq z''_{2N+r}$. We claim that $Z'' \in \widetilde{\mathcal{M}}_{a,b;n}^N$.

All that needs to be checked is that no entry of Z'' is repeated more than twice. Note that it could happen that $k \geq 3$ and $z_{k-2} = z_{k-1} = z_k + 1$, in which case the entry z_{k-1} would be repeated three times in Z'' . Similarly, it could happen that $l + 2 \leq 2N + r$ and $z_l - 1 = z_{l+1} = z_{l+2}$, in which case the entry z_{l+1} would be repeated three times in Z'' . Hence, we must show that these two situations cannot occur.

First assume, if possible, that $k \geq 3$ and $z_{k-2} = z_{k-1} = z_k + 1$. Now, we have $z'_{k-2} = z_{k-2} = z_{k-1} = z'_k$. Since no entry of Z' is repeated more than twice, this implies $z'_k < z'_{k-1}$. But, we have $z_k < z'_k$ and so $z_k + 1 < z'_k + 1 \leq z'_{k-1} = z_{k-1}$, contradicting our assumption. Next assume, if possible, that $l + 2 \leq 2N + r$ and $z_l - 1 = z_{l+1} = z_{l+2}$. Since $z_l > z'_l \geq z'_{l+1} \geq z_{l+1}$, this would imply $z'_l = z'_{l+1} =$

z_{l+1} . Since $z_1 + \dots + z_{l+2} \leq z'_1 + \dots + z'_{l+2}$, we have $z_{l+2} \leq z'_{l+2}$. This yields $z'_{l+1} = z_{l+1} = z_{l+2} \leq z'_{l+2}$ and so $z'_l = z'_{l+1} = z'_{l+2}$, contradicting the fact that no entry of Z' is repeated more than twice. This shows that $Z'' \in \widetilde{\mathcal{M}}_{a,b;n}^N$. We clearly have $Z \trianglelefteq Z'' \trianglelefteq Z'$. Since Z, Z' are assumed to be adjacent, we conclude that $Z' = Z''$, as required. \square

Remark 4.2. With somewhat more effort, one can show that the statement of Lemma 4.1 holds for adjacent multisets in $\mathcal{M}_{1,r;n}^N$, but we shall not need this here.

We can now complete the proof of Lemma 3.11 in this case, as follows. As already mentioned, we shall prove the desired implication for the larger class of multisets $\widetilde{\mathcal{M}}_{a,b;n}^N$. So let $Z, Z' \in \widetilde{\mathcal{M}}_{a,b;n}^N$ be such that $Z \trianglelefteq Z'$. Let t be a sufficiently large integer such that both \overline{Z} and \overline{Z}' are defined and in $\widetilde{\mathcal{M}}_{a,b;n}^{t+1-N-r}$; see Section 3. Note that this definition does indeed work for multisets in $\widetilde{\mathcal{M}}_{a,b;n}^N$. Thus, \overline{Z} and \overline{Z}' are obtained by taking the complements of the multisets

$$\{t - z \mid z \in Z\} \quad \text{and} \quad \{t - z' \mid z' \in Z'\}$$

in the multiset $\{0, 0, 1, 1, 2, 2, \dots, t, t\}$. We must show that $\overline{Z}' \trianglelefteq \overline{Z}$. For this purpose, we can assume without loss of generality that Z, Z' are adjacent with respect to \trianglelefteq . Then Lemma 4.1 shows that Z is obtained from Z' by increasing a suitable entry by 1 and decreasing another suitable entry by 1. A similar statement then also holds for \overline{Z} and \overline{Z}' and one immediately checks that $\overline{Z}' \trianglelefteq \overline{Z}$. Thus, Lemma 3.11 holds in the case where $b' = 0$.

5. PROOF OF LEMMA 3.11: THE CASE WHERE $b' > 0$

Throughout this section we assume that $b' > 0$. An essential feature of this case is that then all entries in a multiset in $\mathcal{M}_{a,b;n}^N$ are distinct; that is, we are dealing with actual sets (finite subsets of $\mathbb{Z}_{\geq 0}$) and not just with multisets. Now it might be possible to use a similar argument as in the previous section, but the following example illustrates that adjacent pairs with respect to \trianglelefteq certainly are more difficult to describe in this case.

Example 5.1. Let $n = N = 2$, $a = 2$ and $b = 3$. We have:

(λ, μ)	$Z_{2,3;2}^2(\lambda, \mu)$
$(\emptyset, 11)$	$\{5, 4, 3, 2, 1\}$
$(\emptyset, 2)$	$\{6, 5, 3, 1, 0\}$
$(1, 1)$	$\{7, 4, 3, 1, 0\}$
$(11, \emptyset)$	$\{7, 5, 2, 1, 0\}$
$(2, \emptyset)$	$\{9, 3, 2, 1, 0\}$

In this table, bipartitions in consecutive rows are adjacent with respect to \trianglelefteq (and these are all the adjacent pairs). Thus, we see that adjacent sets $Z, Z' \in \mathcal{M}_{2,3;2}^2$ can differ in more than 2 entries. (This is not an isolated example.)

Because of this difficulty, we follow an alternative route for proving Lemma 3.11.

It will be useful to introduce the following notation concerning finite subsets of $\mathbb{Z}_{\geq 0}$. If X is such a subset, we denote by $\#X$ the number of elements of X and by

X^+ the sum of the entries of X . Let $M \geq 0$ be an integer such that $x \leq \#X + M - 1$ for all $x \in X$. Then we define

$$\widehat{X}_M := \{0, 1, 2, \dots, \#X + M - 1\} \setminus \{\#X + M - 1 - x \mid x \in X\}.$$

Finally, given two finite subsets $X, Y \subseteq \mathbb{Z}_{\geq 0}$ such that $\#X = \#Y$ and $X^+ = Y^+$, it makes sense to define $X \trianglelefteq Y$. (Just arrange the entries of X and Y in decreasing order and argue as in Remark 3.3.)

It is well-known that these definitions can be interpreted in terms of partitions. Let us briefly recall how this works. Let λ be a partition of some integer $k \geq 0$. Let $m \geq 0$ be a (large) integer and write

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_m \geq 0).$$

The corresponding β -set is defined as

$$B_m(\lambda) := \{\lambda_i + m - i \mid 1 \leq i \leq m\}.$$

Thus, $B_m(\lambda)$ is a finite subset of $\mathbb{Z}_{\geq 0}$ such that $\#B_m(\lambda) = m$ and $B_m(\lambda)^+ = k + \binom{m}{2}$. Conversely, given a finite subset $X \subseteq \mathbb{Z}_{\geq 0}$, with m elements and such that $X^+ = k + \binom{m}{2}$ where $k \geq 0$, there exists a unique partition λ of k such that $X = B_m(\lambda)$. Thus, every finite subset $X \subseteq \mathbb{Z}_{\geq 0}$ can be regarded as the β -set of a partition of a suitable integer $k \geq 0$. One immediately checks that

$$\lambda \trianglelefteq \mu \quad \Leftrightarrow \quad B_m(\lambda) \trianglelefteq B_m(\mu)$$

for all partitions λ, μ of k . Furthermore, given a finite subset $X \subseteq \mathbb{Z}_{\geq 0}$ and an integer $M \geq 0$ as above, the set \widehat{X}_M has the following interpretation. Write $X = B_m(\lambda)$ where $m = \#X$ and λ is a partition. Then, by [22, I.1.7], we have

$$\widehat{X}_M = \{0, 1, \dots, m + M - 1\} \setminus \{m + M - 1 - x \mid x \in X\} = B_M(\overline{\lambda}),$$

where $\overline{\lambda}$ is the conjugate partition.

Lemma 5.2. *Let $X, Y \subseteq \mathbb{Z}_{\geq 0}$ be two finite subsets such that $\#X = \#Y$ and $X^+ = Y^+$. Let $M \geq 0$ be an integer such that $x \leq \#X + M - 1$ for all $x \in X$ and $y \leq \#Y + M - 1$ for all $y \in Y$. Then we have*

$$X \trianglelefteq Y \quad \Rightarrow \quad \widehat{Y}_M \trianglelefteq \widehat{X}_M.$$

Proof. Let us write $X = B_m(\lambda)$ and $Y = B_m(\mu)$ where $m \geq 0$ and λ, μ are partitions of the same integer. Since $X \trianglelefteq Y$, we have $\lambda \trianglelefteq \mu$. By [22, I.1.11], we then also have $\overline{\mu} \trianglelefteq \overline{\lambda}$ (where $\overline{\lambda}$ and $\overline{\mu}$ are the conjugate partitions). Furthermore, as explained above, we have $B_M(\overline{\lambda}) = \widehat{X}_M$ and $B_M(\overline{\mu}) = \widehat{Y}_M$. Since $\overline{\mu} \trianglelefteq \overline{\lambda}$, we then also have $\widehat{Y}_M \trianglelefteq \widehat{X}_M$. \square

If $a = 2$ and $b' = 1$, then the above result immediately implies that Lemma 3.11 holds. (Just note that \overline{Z} , as defined in Section 3, is nothing but \widehat{Z}_M in this case, for suitable values of t, M .) In order to deal with the general case, we need the following result.

Lemma 5.3. *Let λ and μ be partitions of k . Let $l \geq 0$ be an integer. Denote by λ' the partition obtained by adding l as a new part to λ . Similarly, let μ' be the partition obtained by adding l as a new part to μ . (Thus, λ' and μ' are partitions of $k + l$.) Then we have $\lambda \trianglelefteq \mu$ if and only if $\lambda' \trianglelefteq \mu'$.*

Proof. We note that the operation of adding l to λ is equivalent to increasing the largest l entries of the conjugate partition $\bar{\lambda}$ by 1; a similar statement also holds for μ . Now assume that $\lambda \leq \mu$. Then we have $\bar{\mu} \leq \bar{\lambda}$ by [22, I.1.11]. One immediately checks that increasing the largest l entries of $\bar{\lambda}$ and $\bar{\mu}$ does not effect the dominance order relation; that is, we also have $\bar{\mu}' \leq \bar{\lambda}'$. But then [22, I.1.11] implies again that $\lambda' \leq \mu'$, as desired. The argument for the reverse implication is analogous. \square

Corollary 5.4. *Let $X, Y, U \subseteq \mathbb{Z}_{\geq 0}$ be finite subsets such that $\#X = \#Y$, $X^+ = Y^+$ and $U \cap X = U \cap Y = \emptyset$. Then we have*

$$X \leq Y \quad \Rightarrow \quad U \cup X \leq U \cup Y.$$

Proof. The assertion follows by applying Lemma 5.3 repeatedly. \square

We can now complete the proof of Lemma 3.11, as follows. Let $Z, Z' \in \mathcal{M}_{a,b;n}^N$ be such that $Z \leq Z'$. Let t be a sufficiently large integer such that both \bar{Z} and \bar{Z}' are defined and in $\mathcal{M}_{a,b;n}^{t+1-N-r}$; see Section 3. Thus, \bar{Z} and \bar{Z}' are obtained by taking the complements of

$$\{ta + b' - z \mid z \in Z\} \quad \text{and} \quad \{ta + b' - z' \mid z' \in Z'\}$$

in the set $Z_{a,b'}^t = \{0, a, 2a, \dots, ta, b', a + b', 2a + b', \dots, ta + b'\}$. Let U be the set of all $i \in \{0, 1, 2, 3, \dots, ta + b'\}$ such that $i \not\equiv 0 \pmod{a}$ and $i \not\equiv b' \pmod{a}$. Then

$$\{0, 1, 2, 3, \dots, ta + b'\} = U \cup Z_{a,b'}^t \quad (\text{disjoint union}).$$

(In the special case where $a = 2$ and $b' = 1$, we have $U = \emptyset$.) Clearly, we also have

$$\begin{aligned} \bar{Z} &= \{0, 1, 2, 3, \dots, ta + b'\} \setminus (U \cup \{ta + b' - z \mid z \in Z\}), \\ \bar{Z}' &= \{0, 1, 2, 3, \dots, ta + b'\} \setminus (U \cup \{ta + b' - z' \mid z' \in Z'\}). \end{aligned}$$

Now note that $U = \{ta + b' - u \mid u \in U\}$. This yields

$$\begin{aligned} \bar{Z} &= \{0, 1, 2, 3, \dots, ta + b'\} \setminus \{ta + b' - x \mid x \in U \cup Z\} = (\widehat{U \cup Z})_M, \\ \bar{Z}' &= \{0, 1, 2, 3, \dots, ta + b'\} \setminus \{ta + b' - x' \mid x' \in U \cup Z'\} = (\widehat{U \cup Z'})_M, \end{aligned}$$

where $M = ta + b' - \#(U \cup Z) + 1$. Now, since $Z \leq Z'$, we also have $U \cup Z \leq U \cup Z'$; see Corollary 5.4. Then Lemma 5.2 yields that

$$\bar{Z}' = (\widehat{U \cup Z'})_M \leq (\widehat{U \cup Z})_M = \bar{Z},$$

as desired. This completes the proof of Lemma 3.11.

Example 5.5. Let $n = 3$, $a = 3$ and $b = 1$. Then $r = 0$, $b' = 1$. Consider the bipartitions $(\lambda, \mu) = (11, 1)$ and $(\lambda', \mu') = (21, \emptyset)$. Choosing $N = 2$, we have

$$Z := Z_{3,1;3}^2(\lambda, \mu) = \{7, 6, 4, 0\} \quad \text{and} \quad Z' := Z_{3,1;3}^2(\lambda', \mu') = \{10, 4, 3, 0\}.$$

We see that $Z \leq Z'$, that is, $(11, 1) \preceq_{3,1} (21, \emptyset)$. Choosing $t = 3$, we have $ta + b' = 10$. Considering the appropriate complements in $\{0, 3, 6, 9, 1, 4, 7, 10\}$ we obtain

$$\bar{Z} = \{9, 7, 1, 0\} \quad \text{and} \quad \bar{Z}' = \{9, 4, 3, 1\}.$$

Thus, we have $\bar{Z}' \leq \bar{Z}$, as required. Now let us see how this fits with the above argument. We have $U = \{8, 5, 2\} = \{10 - 8, 10 - 5, 10 - 2\}$. This yields

$$U \cup Z = \{8, 7, 6, 5, 4, 2, 0\} \quad \text{and} \quad U \cup Z' = \{10, 8, 5, 4, 3, 2, 0\}.$$

Now $\#(U \cup Z) = \#(U \cup Z') = 7$ and $M = ta + b' - \#(U \cup Z) + 1 = 4$. This yields the required equalities $\overline{Z} = (\widehat{U \cup Z})_4$ and $\overline{Z}' = (\widehat{U \cup Z'})_4$.

6. EXAMPLE: THE ASYMPTOTIC CASE

Throughout this section, we shall assume that $a, b \in \mathbb{Z}$ are such that

$$b > (n-1)a > 0.$$

This is the “asymptotic case” considered by Bonnafé–Iancu [1], [3]. We shall see that, in this case, the pre-order relation $\preceq_{a,b}$ admits a more direct and familiar description, without reference to the parameters a and b . The proof of this description would simplify drastically if b were assumed to be very large with respect to a (e.g., $b > 2na$). Assuming only that $b > (n-1)a > 0$ requires a careful analysis of the arrangement of the entries of $Z_{a,b}^N(\lambda, \mu)$.

Let r, b' be as in Section 3. Then $r \geq n-1$ and $0 \leq b' < a$; furthermore, we must have $b' > 0$ if $r = n-1$. Given a partition ν of some integer m , we denote by $\nu_1 \geq \nu_2 \geq \dots \geq 0$ the parts of ν and write $|\nu| = m$.

Lemma 6.1. *Assume that $\lambda = \emptyset$. If $\mu_1 = n$ and $r = n-1$, then the largest entry of $Z_{a,b}^N(\lambda, \mu)$ is $(N+n-1)a$. Otherwise, the largest entry of $Z_{a,b}^N(\lambda, \mu)$ is $(N+r-1)a + b'$.*

Proof. The entries of $Z_{a,b}^N(\lambda, \mu)$ which arise from λ are $(N+r-i)a + b'$ for $1 \leq i \leq N+r$. The largest of these entries is $(N+r-1)a + b'$. On the other hand, the largest entry of $Z_{a,b}^N(\lambda, \mu)$ arising from μ is $(\mu_1 + N-1)a$. This immediately yields the assertion for the case where $\mu_1 = n$ and $r = n-1$. Otherwise, we have $\mu_1 \leq r$ and so $N+r-1 \geq \mu_1 + N-1$. Hence, in this case, the largest entry of $Z_{a,b}^N(\lambda, \mu)$ is $(N+r-1)a + b'$. \square

Remark 6.2. Let $l \geq 1$ be such that $\lambda_l + N+r-l \geq \mu_1 + N-1$. Then, clearly, the largest l entries of $Z_{a,b}^N(\lambda, \mu)$ are $(\lambda_i + N+r-i)a + b'$ for $1 \leq i \leq l$. This simple remark will be used frequently in what follows.

Lemma 6.3. *Assume that $\lambda \neq \emptyset$ and let $l = |\lambda| \geq 1$; choose N large enough to have $l \leq N+r$. Then*

$$\lambda_l + N+r-l \geq \mu_1 + N-1$$

and so the largest l entries of $Z_{a,b}^N(\lambda, \mu)$ are $(\lambda_i + N+r-i)a + b'$ for $1 \leq i \leq l$.

Proof. Since $\mu_1 \leq |\mu| = n - |\lambda| = n - l$ and $r \geq n-1$, we have $\mu_1 + N-1 \leq N+n-l-1 \leq N+r-l \leq \lambda_l + N+r-l$, as required. The statement about the largest l entries of $Z_{a,b}^N(\lambda, \mu)$ follows by Remark 6.2. \square

Definition 6.4. The dominance order between bipartitions (λ, μ) and (λ', μ') of n is defined by

$$(\lambda, \mu) \leq (\lambda', \mu') \stackrel{\text{def}}{\iff} \begin{cases} \sum_{1 \leq i \leq d} \lambda_i \leq \sum_{1 \leq i \leq d} \lambda'_i & \text{for all } d, \\ |\lambda| + \sum_{1 \leq i \leq d} \mu_i \leq |\lambda'| + \sum_{1 \leq i \leq d} \mu'_i & \text{for all } d, \end{cases}$$

where $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq 0)$, $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \dots \geq 0)$, $\mu = (\mu_1 \geq \mu_2 \geq \dots \geq 0)$ and $\mu' = (\mu'_1 \geq \mu'_2 \geq \dots \geq 0)$. (This appeared in the work of Dipper–James–Murphy [7].)

Remark 6.5. Let us define the dominance order also for partitions of possibly different numbers (as, for example, in the section on “raising operators” in [22, I.1]). Then we have:

$$(\lambda, \mu) \trianglelefteq (\lambda', \mu') \quad \Leftrightarrow \quad \lambda \trianglelefteq \lambda' \quad \text{and} \quad \overline{\mu'} \trianglelefteq \overline{\mu}.$$

This easily follows by using a slight variation of the argument in [22, I.1.11].

Remark 6.6. Let (λ, μ) and (λ', μ') be bipartitions of n such that $(\lambda, \mu) \trianglelefteq (\lambda', \mu')$ and $(\lambda, \mu) \neq (\lambda', \mu')$. Assume that (λ, μ) and (λ', μ') are adjacent with respect to \trianglelefteq (that is, if $(\lambda, \mu) \trianglelefteq (\lambda'', \mu'') \trianglelefteq (\lambda', \mu')$ for some bipartition (λ'', μ'') , then $(\lambda, \mu) = (\lambda'', \mu'')$ or $(\lambda'', \mu'') = (\lambda', \mu')$). Then we are in one of the following three cases.

- (a) $\mu = \mu'$ and there exist indices $i < j$ such that λ' is obtained from λ by increasing λ_i by 1 and decreasing λ_j by 1. (In particular, λ, λ' are adjacent with respect to \trianglelefteq .)
- (b) $\lambda = \lambda'$ and there exist indices $i < j$ such that μ' is obtained from μ by increasing μ_i by 1 and decreasing μ_j by 1. (In particular, μ, μ' are adjacent with respect to \trianglelefteq .)
- (c) $|\lambda| < |\lambda'|$ and there exist indices i, j such that λ' is obtained from λ by increasing λ_i by 1 and μ' is obtained from μ by decreasing μ_j by 1.

This follows easily by an argument similar to [22, I.1.16].

Proposition 6.7. *Let (λ, μ) and (λ', μ') be bipartitions of n . The following holds*

$$(\lambda, \mu) \trianglelefteq_{a,b} (\lambda', \mu') \quad \Rightarrow \quad \lambda \trianglelefteq \lambda'.$$

Proof. If $\lambda = \emptyset$, then this is clear. Now assume that $\lambda' = \emptyset$. We show that then we also have $\lambda = \emptyset$. Assume that this is not the case. Then, by Lemma 6.3, the largest entry of $Z_{a,b}^N(\lambda, \mu)$ is $(\lambda_1 + N + r - 1)a + b'$. The condition $Z_{a,b}^N(\lambda, \mu) \trianglelefteq Z_{a,b}^N(\lambda', \mu')$ implies, in particular, that the largest entry of $Z_{a,b}^N(\lambda, \mu)$ is less than or equal to the largest entry of $Z_{a,b}^N(\lambda', \mu')$. Using Lemma 6.1, we distinguish two cases. If $\mu'_1 = n$ and $r = n - 1$, we must have $(\lambda_1 + N + n - 2)a + b' \leq (N + n - 1)a$. Since then also $b' > 0$, this forces that $\lambda_1 = 0$, a contradiction. Otherwise, we have $(\lambda_1 + N + r - 1)a + b' \leq (N + r - 1)a + b'$ which yields again that $\lambda_1 = 0$, a contradiction. Thus, if $\lambda' = \emptyset$, then $\lambda = \emptyset$, as required.

We can now assume that $\lambda \neq \emptyset$ and $\lambda' \neq \emptyset$. Let

$$k = \max\{i \geq 1 \mid \lambda_i > 0\} \quad \text{and} \quad k' = \max\{i \geq 1 \mid \lambda'_i > 0\}.$$

Clearly, we have $k \leq |\lambda|$ and $k' \leq |\lambda'|$. So, by Lemma 6.3, we have

$$\lambda_k + N + r - k \geq \mu_1 + N - 1 \quad \text{and} \quad \lambda'_{k'} + N + r - k' \geq \mu'_1 + N - 1;$$

also the largest k entries of $Z_{a,b}^N(\lambda, \mu)$ are $(\lambda_i + N + r - i)a + b$ for $1 \leq i \leq k$, while the largest k' entries of $Z_{a,b}^N(\lambda', \mu')$ are $(\lambda'_i + N + r - i)a + b'$ for $1 \leq i \leq k'$.

Now we distinguish two cases. First case: we also have $\lambda'_k + N + r - k \geq \mu'_1 + N - 1$. Let d be such that $1 \leq d \leq k$; then, by Remark 6.2, the sum of the largest d entries of $Z_{a,b}^N(\lambda, \mu)$ is

$$\sum_{1 \leq i \leq d} ((\lambda_i + N + r - i)a + b),$$

while the sum of the largest d entries of $Z_{a,b}^N(\lambda', \mu')$ is

$$\sum_{1 \leq i \leq d} ((\lambda'_i + N + r - i)a + b').$$

Now the condition $Z_{a,b}^N(\lambda, \mu) \trianglelefteq Z_{a,b}^N(\lambda', \mu')$ implies, in particular, that the first sum is less than or equal to the second sum. Hence, we obtain

$$\sum_{1 \leq i \leq d} \lambda_i \leq \sum_{1 \leq i \leq d} \lambda'_i, \quad \text{for all } d \in \{1, \dots, k\}$$

and so $\lambda \trianglelefteq \lambda'$, as desired.

It remains to consider the second case: where we have

$$\lambda'_k + N + r - k < \mu'_1 + N - 1.$$

Note that, since $\lambda'_{k'} + N + r - k' \geq \mu'_1 + N - 1$, this forces $k > k'$. So there exists an index $l \in \{k' + 1, \dots, k\}$ such that

$$\begin{aligned} \lambda'_i + N + r - i &\geq \mu'_1 + N - 1 & \text{for } i = k', k' + 1, \dots, l - 1, \\ \lambda'_l + N + r - l &< \mu'_1 + N - 1. \end{aligned}$$

Consequently, the l largest entries of $Z_{a,b}^N(\lambda', \mu')$ are

$$(\lambda'_i + N + r - i)a + b' \quad (1 \leq i \leq l - 1) \quad \text{and} \quad (\mu'_1 + N - 1)a.$$

Since $l \leq k$, the l largest entries of $Z_{a,b}^N(\lambda, \mu)$ are $(\lambda_i + N + r - i)a + b'$ for $1 \leq i \leq l$. Taking the sums of these entries, the condition $Z_{a,b}^N(\lambda, \mu) \trianglelefteq Z_{a,b}^N(\lambda', \mu')$ implies, in particular, that

$$\sum_{1 \leq i \leq l} ((\lambda_i + N + r - i)a + b') \leq (\mu'_1 + N - 1)a + \sum_{1 \leq i \leq l-1} ((\lambda'_i + N + r - i)a + b').$$

Hence, we obtain

$$(\lambda_l + r - l)a + b' + \left(\sum_{1 \leq i \leq l-1} \lambda_i \right) a \leq (\mu'_1 - 1)a + \left(\sum_{1 \leq i \leq l-1} \lambda'_i \right) a$$

and so

$$ra + b' - la + \left(\sum_{1 \leq i \leq l} \lambda_i \right) a \leq \mu'_1 a - a + |\lambda'|a.$$

Finally, since, $\mu'_1 \leq |\mu'|$, we deduce that

$$ra + b' \leq \mu'_1 a - a + |\lambda'|a + \left(l - \sum_{1 \leq i \leq l} \lambda_i \right) a \leq (n - 1)a + \left(l - \sum_{1 \leq i \leq l} \lambda_i \right) a.$$

Since $l \leq k$, we have $\lambda_i \geq 1$ for $1 \leq i \leq l$. Hence, the right hand side of the above inequality is less than or equal to $(n - 1)a$. We conclude that $b = ra + b' \leq (n - 1)a$, a contradiction. \square

Corollary 6.8. *Recall that $b > (n - 1)a > 0$. Let (λ, μ) and (λ', μ') be bipartitions of n . Then we have*

$$(\lambda, \mu) \preceq_{a,b} (\lambda', \mu') \quad \Leftrightarrow \quad (\lambda, \mu) \trianglelefteq (\lambda', \mu').$$

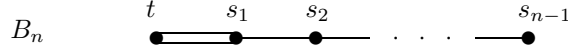
In particular, $\preceq_{a,b}$ is not just a pre-order but a partial order.

Proof. Assume first that $(\lambda, \mu) \preceq_{a,b} (\lambda', \mu')$. Then, by Proposition 3.12, we know that we also have $(\overline{\mu}', \overline{\lambda}') \preceq_{a,b} (\overline{\mu}, \overline{\lambda})$. Now Proposition 6.7 implies that $\lambda \trianglelefteq \lambda'$ and $\overline{\mu}' \trianglelefteq \overline{\mu}$, as desired.

Conversely, assume that $(\lambda, \mu) \trianglelefteq (\lambda', \mu')$. In order to show that $(\lambda, \mu) \preceq_{a,b} (\lambda', \mu')$, we may assume without loss of generality that $(\lambda, \mu) \neq (\lambda', \mu')$ and that (λ, μ) and (λ', μ') are adjacent with respect to \trianglelefteq . Thus, we are in one of the three cases in Remark 6.6. In each of these cases, one can check directly that $(\lambda, \mu) \preceq_{a,b} (\lambda', \mu')$. We omit the details as this will also follow by combining Theorem 7.11 with the implication $(*)$ in Example 8.1 below. \square

7. THE PRE-ORDER RELATION \preceq_L IN TYPE B_n

Throughout this section, let $n \geq 1$ and W_n be a Coxeter group of type B_n , with generators and diagram given by



As in Example 2.12, we write $\text{Irr}(W_n) = \{E^{(\lambda, \mu)} \mid (\lambda, \mu) \text{ is a bipartition of } n\}$. A weight function $L: W_n \rightarrow \mathbb{Z}$ is specified by the two parameters $b := L(t)$ and $a := L(s_i)$ for $1 \leq i \leq n-1$. From now until Example 7.13 (at the very end of this section), we shall assume that

$$b \geq 0 \text{ and } a > 0.$$

(It is convenient to allow the possibility that $b = 0$ because this is related to groups of type D_n ; see Example 8.6 in the following section.) The main result of this section is Theorem 7.11, which provides a combinatorial description of the pre-order relation \preceq_L on $\text{Irr}(W_n)$ in terms of the pre-orders on bipartitions considered in Section 3.

We begin by collecting some known results. The effect of tensoring with the sign representation is given as follows.

Lemma 7.1. *Let (λ, μ) be a bipartition of n . Then $E^{(\lambda, \mu)} \otimes \varepsilon = E^{(\overline{\mu}, \overline{\lambda})}$.*

Proof. This can be found, for example, in [17, 5.5.6]. \square

Proposition 7.2 (Lusztig [21, 22.14]). *Let (λ, μ) be a bipartition of n . Then*

$$a_{E^{(\lambda, \mu)}} = a_{a,b}(\lambda, \mu); \quad \text{see Definition 3.8.}$$

(Note that Lusztig assumes that $b > 0$ but this formula also works for $b = 0$.)

Remark 7.3. By Remark 2.2 and Lemma 7.1, we have

$$\omega_L(E^{(\lambda, \mu)}) = a_{a,b}(\overline{\mu}, \overline{\lambda}) - a_{a,b}(\lambda, \mu).$$

Using [17, Lemmas 6.2.6 and 6.2.8], we obtain the following more direct formula:

$$\omega_L(E^{(\lambda, \mu)}) = (|\lambda| - |\mu|)b + 2(n(\overline{\lambda}) - n(\lambda) + n(\overline{\mu}) - n(\mu))a,$$

where $n(\nu)$ (for any partition ν) is defined as in Example 2.11.

Using the combinatorics in Section 3, Lusztig has determined the families of $\text{Irr}(W_n)$. It turns out that these are precisely given by the “combinatorial families” in Definition 3.4.

Proposition 7.4 (Lusztig [21, 23.1]). *Let (λ, μ) and (λ', μ') be bipartitions of n . Then:*

$$E^{(\lambda, \mu)}, E^{(\lambda', \mu')} \text{ belong to the same family} \iff Z_{a,b}^N(\lambda, \mu) = Z_{a,b}^N(\lambda', \mu'),$$

where $N \geq 0$ is a sufficiently large integer (see Definition 3.1).

Let us now turn to the description of the pre-order relation \preceq_L . To state the following result, we recall that the maximal parabolic subgroups of W_n are of the form $W_k \times H_l$ where $n = k + l$ ($k \geq 0, l \geq 1$). Here, W_k is of type B_k (generated by t, s_1, \dots, s_{k-1}) and H_l is of type A_{l-1} (generated by $s_{k+1}, s_{k+2}, \dots, s_{n-1}$). It is understood that $W_0 = H_1 = \{1\}$. Let ε_l denote the sign representation of H_l .

Lemma 7.5 (Cf. Spaltenstein [23, §3]). *Let $E, E' \in \text{Irr}(W_n)$. Then $E \preceq_L E'$ if and only if there exists a sequence $E = E_0, E_1, \dots, E_m = E'$ in $\text{Irr}(W_n)$ such that, for each $i \in \{1, 2, \dots, m\}$, the following condition is satisfied:*

There exists a decomposition $n = k_i + l_i$ ($k_i \geq 0, l_i \geq 1$) and $M_i, M'_i \in \text{Irr}(W_{k_i})$ where $M_i \preceq_L M'_i$ within $\text{Irr}(W_{k_i})$, such that either

$$M_i \boxtimes \varepsilon_{l_i} \uparrow E_{i-1} \quad \text{and} \quad M'_i \boxtimes \varepsilon_{l_i} \rightsquigarrow_L E_i$$

or

$$M_i \boxtimes \varepsilon_{l_i} \uparrow E_i \otimes \varepsilon \quad \text{and} \quad M'_i \boxtimes \varepsilon_{l_i} \rightsquigarrow_L E_{i-1} \otimes \varepsilon.$$

(Here, $M_i \boxtimes \varepsilon_{l_i}$ and $M'_i \boxtimes \varepsilon_{l_i}$ are representations of $W_{k_i} \times H_{l_i} \subseteq W_n$.)

Proof. The “if” part is clear by the definition of \preceq_L . To prove the “only if” part, it is sufficient to consider an elementary step in Definition 2.5. That is, we can assume that there is a subset $I \subsetneq S$ and $M_1, M'_1 \in \text{Irr}(W_I)$, where $M_1 \preceq_L M'_1$ within $\text{Irr}(W_I)$, such that one of the following two conditions holds.

- $M_1 \uparrow E$ and $M'_1 \rightsquigarrow_L E'$.
- $M_1 \uparrow E' \otimes \varepsilon$ and $M'_1 \rightsquigarrow_L E \otimes \varepsilon$.

By Remark 2.9, we can further assume that W_I is a maximal parabolic subgroup of W_n , that is, we have $W_I = W_k \times H_l$ where $k \geq 0, l \geq 1$. Since $H_l \cong \mathfrak{S}_l$, we can write

$$M_1 = \tilde{M}_1 \boxtimes E^\lambda \quad \text{and} \quad M'_1 = \tilde{M}'_1 \boxtimes E^\mu$$

where $\tilde{M}_1, \tilde{M}'_1 \in \text{Irr}(W_k)$ and λ, μ are partitions of l . By Remark 2.8 and Example 2.11, we have

$$\tilde{M}_1 \preceq_L \tilde{M}'_1 \quad (\text{within } \text{Irr}(W_k)) \quad \text{and} \quad \lambda \trianglelefteq \mu.$$

Let $H_{\bar{\mu}} \subseteq H_l$ be the parabolic subgroup corresponding to the Young subgroup $\mathfrak{S}_{\bar{\mu}} \subseteq \mathfrak{S}_l$. By Example 2.11 (see the proof of the implication “(b) \Rightarrow (c)”), we have $\varepsilon_{\bar{\mu}} \uparrow E^\lambda$ and $\varepsilon_{\bar{\mu}} \rightsquigarrow_L E^\mu$ where $\varepsilon_{\bar{\mu}}$ is the sign representation of $H_{\bar{\mu}}$. Hence, by Remark 2.8 and the transitivity of induction, one of the following two conditions holds.

- $\tilde{M}_1 \boxtimes \varepsilon_{\bar{\mu}} \uparrow E$ and $\tilde{M}'_1 \boxtimes \varepsilon_{\bar{\mu}} \rightsquigarrow_L E'$.
- $\tilde{M}_1 \boxtimes \varepsilon_{\bar{\mu}} \uparrow E' \otimes \varepsilon$ and $\tilde{M}'_1 \boxtimes \varepsilon_{\bar{\mu}} \rightsquigarrow_L E \otimes \varepsilon$.

Thus, we have replaced the maximal parabolic subgroup $W_I = W_k \times H_l$ (that we started with) by the parabolic subgroup $W_k \times H_{\bar{\mu}}$, where we consider the sign representation on the $H_{\bar{\mu}}$ -factor. We will now embed $W_k \times H_{\bar{\mu}}$ into a different maximal parabolic subgroup such that the required conditions will be satisfied.

For this purpose, let $\bar{\mu} = (\bar{\mu}_1 \geq \bar{\mu}_2 \geq \dots \geq \bar{\mu}_d \geq 1)$ be the non-zero parts of $\bar{\mu}$. Correspondingly, we have a direct product decomposition

$$H_{\bar{\mu}} = H_{\bar{\mu}_1} \times H_{\bar{\mu}_2} \times \dots \times H_{\bar{\mu}_d}.$$

Grouping the first $d - 1$ factors together, we obtain

$$W_k \times H_{\bar{\mu}} \subseteq (W_k \times H_{l_1}) \times H_{l'} \subseteq W_{k'} \times H_{l'}$$

where $l_1 := \bar{\mu}_1 + \bar{\mu}_2 + \dots + \bar{\mu}_{d-1}$, $k' := k + l_1$ and $l' := \bar{\mu}_d$. Using Remarks 2.8 and 2.9, we conclude that there exist $M, M' \in \text{Irr}(W_{k'})$ such that

$$(*) \quad \tilde{M}_1 \boxtimes \varepsilon_{\bar{\mu}} \uparrow M \boxtimes \varepsilon_{l'}, \quad \tilde{M}'_1 \boxtimes \varepsilon_{\bar{\mu}} \rightsquigarrow_L M' \boxtimes \varepsilon_{l'}$$

and one of the following two conditions holds.

- $M \boxtimes \varepsilon_{l'} \uparrow E$ and $M' \boxtimes \varepsilon_{l'} \rightsquigarrow_L E'$.
- $M \boxtimes \varepsilon_{l'} \uparrow E' \otimes \varepsilon$ and $M' \boxtimes \varepsilon_{l'} \rightsquigarrow_L E \otimes \varepsilon$.

Finally, we use Remark 2.8 to conclude that $M \preceq_L M'$ within $\text{Irr}(W_{k'})$. Indeed, since $\tilde{M}_1 \preceq_L \tilde{M}'_1$ within $\text{Irr}(W_k)$, we have $\tilde{M}_1 \boxtimes \varepsilon_{\bar{\mu}} \preceq_L \tilde{M}'_1 \boxtimes \varepsilon_{\bar{\mu}}$ within $\text{Irr}(W_k \times H_{\bar{\mu}})$. Then $(*)$ implies that $M \boxtimes \varepsilon_{l'} \preceq_L M' \boxtimes \varepsilon_{l'}$ within $\text{Irr}(W_{k'} \times H_{l'})$ and, hence, $M \preceq_L M'$ within $\text{Irr}(W_{k'})$, as required. \square

In order to proceed, we need some more precise information about the induction of representations from $W_k \times H_l$ to W_n . The basic tool is the following rule:

Lemma 7.6 (Pieri's Rule for W_n). *Let $n = k + l$ where $k \geq 0$, $l \geq 1$. Let (α, β) be a bipartition of k and (λ, μ) be a bipartition of n . Then we have*

$$E^{(\alpha, \beta)} \otimes \varepsilon_l \uparrow E^{(\lambda, \mu)}$$

if and only if (λ, μ) can be obtained by increasing l parts of (α, β) by 1.

Proof. This can be reduced to a statement about representations of the symmetric group, where it corresponds to the classical ‘‘Pieri Rule’’ for symmetric functions. See [17, 6.1.9] and the proof of [17, 6.4.7] for details. \square

In order to be able to apply this rule in our context, we need to interpret it in terms of multisets. So let $n = k + l$, (α, β) , (λ, μ) be as above and assume that $E^{(\alpha, \beta)} \otimes \varepsilon_l \uparrow E^{(\lambda, \mu)}$. Let $N \geq 0$ be a sufficiently large integer and consider the multisets $Z_{a,b}^N(\alpha, \beta)$ and $Z_{a,b}^N(\lambda, \mu)$. Let

$$Z_{a,b}^N(\alpha, \beta) = \{u_1, \dots, u_{2N+r}\} \quad \text{where} \quad u_1 \geq u_2 \geq \dots \geq u_{2N+r}.$$

Then, by Lemma 7.6, we have

$$Z_{a,b}^N(\lambda, \mu) = \{u_i + \delta_i a \mid 1 \leq i \leq 2N + r\} \quad \text{where} \quad \delta_i \in \{0, 1\};$$

furthermore, the number of $i \in \{1, 2, \dots, 2N + r\}$ such that $\delta_i = 1$ equals l . Note, however, that the entries $u_i + \delta_i a$ are not necessarily arranged in decreasing order! We need to know precisely to what extent this can fail.

We define a sequence $(z_i)_{1 \leq i \leq 2N+r}$ as follows. For all i such that $\delta_i = 0$, $\delta_{i+1} = 1$ and $u_{i+1} + a > u_i$, we set $z_i := u_{i+1} + a$ and $z_{i+1} := u_i$. For all the remaining i , we set $z_i := u_i + \delta_i a$. Thus, we have $Z_{a,b}^N(\lambda, \mu) = \{z_1, \dots, z_{2N+r}\}$.

Example 7.7. Let $k = 14$ and $(\alpha, \beta) = (4311, 32)$.

(a) Let $l = 3$ and $(\lambda, \mu) = (4321, 421)$. Assume that $a = b = 1$. Then $r = 1$, $b' = 0$ and we can take $N = 3$. As in Example 3.2, we obtain

$$\begin{aligned} Z_{1,1}^3(4311, 32) &= \{7, \hat{5}, 5, 3, \hat{2}, 1, \hat{0}\}, \\ Z_{1,1}^3(4321, 421) &= \{7, 6, 5, 3, 3, 1, 1\}, \end{aligned}$$

where the hat indicates that $\delta_i = 1$. Note that, since $u_2 = u_3 = 5$, the sequence (δ_i) is not uniquely determined: we could take either $\delta_2 = 0, \delta_3 = 1$ or $\delta_2 = 1, \delta_3 = 0$. But if we choose the second possibility, then $z_i = u_i + \delta_i$ for all i , and the sequence (z_1, \dots, z_7) is in decreasing order. This will always be the case if $b' = 0$, assuming that whenever we have an equality $u_i = u_{i+1}$, then $\delta_i \geq \delta_{i+1}$.

(b) Let $l = 4$ and $(\lambda, \mu) = (4421, 331)$. Assume that $a = 2, b = 1$. Then $r = 0$, $b' = 1$ and we can take $N = 4$. As in Example 3.2, we obtain

$$\begin{aligned} Z_{2,1}^4(4311, 32) &= \{15, 12, \hat{11}, \hat{8}, \hat{5}, 3, \hat{2}, 0\}, \\ Z_{2,1}^4(4421, 331) &= \{15, 12, 13, 10, 7, 3, 4, 0\}, \end{aligned}$$

where the hat indicates that $\delta_i = 1$. Hence, in this case, we see that some reordering is required. The only critical indices i such that $\delta_i = 0, \delta_{i+1} = 1$ and $u_{i+1} + 2 > u_i$ are $i = 2$ and $i = 6$. Thus, we set $z_2 := u_3 + 2 = 13, z_3 := u_2 = 12, z_6 := u_7 + 2 = 4, z_7 := u_6 = 3$ and $z_i := u_i + \delta_i$ for all the remaining i . Then $(z_1, \dots, z_8) = (15, 13, 12, 10, 7, 4, 3, 0)$ is in decreasing order.

The observations in this example are true in general:

Lemma 7.8. *In the above setting, we have $z_1 \geq z_2 \geq \dots \geq z_{2N+r}$ and*

$$\sum_{1 \leq i \leq d} z_i \leq \min\{d, l\}a + \sum_{1 \leq i \leq d} u_i \quad \text{for } 1 \leq d \leq 2N + r.$$

Proof. Assume that $1 \leq i \leq 2N + r - 1$. We want to show that $z_i \geq z_{i+1}$. From the construction, we see that we have the following possibilities:

$$z_i \in \{u_i, u_{i-1}, u_i + a, u_{i+1} + a\} \quad \text{and} \quad z_{i+1} \in \{u_{i+1}, u_i, u_{i+1} + a, u_{i+2} + a\},$$

where the combination $(z_i, z_{i+1}) = (u_i, u_{i+1} + a)$ only occurs if $\delta_i = 0, \delta_{i+1} = 1, u_i \geq u_{i+1} + a$ and the combination $(z_i, z_{i+1}) = (u_{i+1} + a, u_i)$ only occurs if $\delta_i = 0, \delta_{i+1} = 1, u_{i+1} + a > u_i$. Further note that, since $Z_{a,b}^N(\alpha, \beta)$ satisfies the conditions (M1)–(M3) in Section 3, we have $u_i - u_{i+2} \geq a$ for all i . Hence, for all valid combinations of pairs (z_i, z_{i+1}) as above, we have $z_i \geq z_{i+1}$, as claimed.

Now consider the sum $\sum_{1 \leq i \leq d} z_i$ and note that if $d \geq l$, then the desired inequality holds. Assume now that $d < l$ and let $1 \leq i_1 < i_2 < \dots < i_h \leq 2N + r - 1$ be the critical indices i such that $\delta_i = 0, \delta_{i+1} = 1$ and $u_{i+1} + a > u_i$. Note that $i_{j+1} - i_j \geq 2$ for all j . Now we have two cases:

If $d \notin \{i_1, i_2, \dots, i_h\}$, then (z_1, \dots, z_d) will be equal, up to possibly interchanging consecutive indices, to $(u_1 + \delta_1 a, \dots, u_d + \delta_d a)$. Hence, if we sum over these two sequences, we get the same result and so

$$\sum_{1 \leq i \leq d} z_i = \left(\sum_{1 \leq i \leq d} \delta_i \right) a + \sum_{1 \leq i \leq d} u_i \leq \min\{d, l\}a + \sum_{1 \leq i \leq d} u_i.$$

On the other hand, if $d = i_j$ for some j , then $d - 1 \notin \{i_1, i_2, \dots, i_h\}$ and so the previous case applies to the sum $\sum_{1 \leq i \leq d-1} z_i$. Hence, we have

$$\sum_{1 \leq i \leq d} z_i = z_d + \sum_{1 \leq i \leq d-1} z_i \leq z_d + \min\{d-1, l\}a + \sum_{1 \leq i \leq d-1} u_i.$$

Now $z_d = u_{d+1} + a \leq u_d + a$ and so we obtain again the desired estimation. \square

Lemma 7.9. *Let $n = k + l$ where $k \geq 0$, $l \geq 1$. Let (α, β) be a bipartition of k and (λ, μ) be a bipartition of n . Let $N \geq 0$ be a sufficiently large integer and consider the multisets $Z_{a,b}^N(\alpha, \beta)$ and $Z_{a,b}^N(\lambda, \mu)$. Then the following implication holds:*

$$E^{(\alpha, \beta)} \otimes \varepsilon_l \uparrow E^{(\lambda, \mu)} \quad \Rightarrow \quad Z_{a,b}^N(\lambda, \mu) \leq \hat{Z}_{a,b}^N(\alpha, \beta),$$

where $\hat{Z}_{a,b}^N(\alpha, \beta) \in \mathcal{M}_{a,b;n}^N$ is the multiset obtained by increasing the largest l entries of $Z_{a,b}^N(\alpha, \beta)$ by a .

Proof. Write $Z_{a,b}^N(\alpha, \beta) = \{u_1, \dots, u_{2N+r}\}$ and $Z_{a,b}^N(\lambda, \mu) = \{z_1, \dots, z_{2N+r}\}$ as above, where the entries in both multisets are in decreasing order. We have

$$\hat{Z}_{a,b}^N(\alpha, \beta) = \{\hat{u}_1, \dots, \hat{u}_{2N+r}\} \quad \text{where} \quad \hat{u}_i = \begin{cases} u_i + a & \text{for } 1 \leq i \leq l, \\ u_i & \text{for } i > l. \end{cases}$$

Hence, using the estimation in Lemma 7.8, we obtain

$$\sum_{1 \leq i \leq d} z_i \leq \min\{d, l\}a + \sum_{1 \leq i \leq d} u_i = \sum_{1 \leq i \leq d} \hat{u}_i$$

for $1 \leq d \leq 2N + r$. This means that $Z_{a,b}^N(\lambda, \mu) \leq \hat{Z}_{a,b}^N(\alpha, \beta)$, as required. \square

Lemma 7.10 (Lusztig [21, 22.17]). *In the setting of Lemma 7.9, we have*

$$E^{(\alpha, \beta)} \otimes \varepsilon_l \rightsquigarrow_L E^{(\lambda, \mu)} \quad \Rightarrow \quad Z_{a,b}^N(\lambda, \mu) = \hat{Z}_{a,b}^N(\alpha, \beta).$$

Proof. Let us write $\hat{Z}_{a,b}^N(\alpha, \beta) = \{\hat{u}_1, \dots, \hat{u}_{2N+r}\}$ as in the above proof. Then

$$\begin{aligned} \sum_{1 \leq i \leq 2N+r} (i-1)\hat{u}_i &= \sum_{1 \leq i \leq l} (i-1)a + \sum_{1 \leq i \leq 2N+r} (i-1)u_i \\ &= \binom{l}{2}a + \sum_{1 \leq i \leq 2N+r} (i-1)u_i. \end{aligned}$$

Thus, we also have

$$\mathbf{a}_{a,b}(\hat{Z}_{a,b}^N(\alpha, \beta)) = \binom{l}{2}a + \mathbf{a}_{a,b}(Z_{a,b}^N(\alpha, \beta)).$$

By Remark 2.8, Proposition 7.2 and Example 2.11, the right hand side equals $\mathbf{a}_{E^{(\alpha, \beta)} \otimes \varepsilon_l}$. Hence, by Lemma 7.9, the assumption implies that

$$Z_{a,b}^N(\lambda, \mu) \leq \hat{Z}_{a,b}^N(\alpha, \beta) \quad \text{and} \quad \mathbf{a}_{a,b}(Z_{a,b}^N(\lambda, \mu)) = \mathbf{a}_{a,b}(\hat{Z}_{a,b}^N(\alpha, \beta)).$$

So Remark 3.9 shows that $Z_{a,b}^N(\lambda, \mu) = \hat{Z}_{a,b}^N(\alpha, \beta)$, as desired. \square

Now we can state the main result of this section.

Theorem 7.11. *Recall that $b \geq 0$ and $a > 0$. Let (λ, μ) and (λ', μ') be bipartitions of n . Then we have:*

$$E^{(\lambda, \mu)} \leq_L E^{(\lambda', \mu')} \quad \Rightarrow \quad (\lambda, \mu) \preceq_{a,b} (\lambda', \mu') \quad (\text{see Definition 3.4}).$$

Proof. We proceed by induction on n . If $n = 1$, then $(1, \emptyset)$ and $(\emptyset, 1)$ are the only bipartitions of n and the assertion is easily checked directly. Now let $n \geq 2$. We use the characterisation of \preceq_L in Lemma 7.5. It is sufficient to consider an elementary step in that characterisation, that is, we can assume that there exists a decomposition $n = k + l$ ($k \geq 0, l \geq 1$) and $M, M' \in \text{Irr}(W_k)$ where $M \preceq_L M'$ within $\text{Irr}(W_k)$, such that one of the following conditions is satisfied:

- (I) $M \boxtimes \varepsilon_l \uparrow E^{(\lambda, \mu)}$ and $M' \boxtimes \varepsilon_l \rightsquigarrow_L E^{(\lambda', \mu')}$.
- (II) $M \boxtimes \varepsilon_l \uparrow E^{(\lambda', \mu')} \otimes \varepsilon$ and $M' \boxtimes \varepsilon_l \rightsquigarrow_L E^{(\lambda, \mu)} \otimes \varepsilon$.

Now write $M = E^{(\alpha, \beta)}$ and $M' = E^{(\alpha', \beta')}$ where (α, β) and (α', β') are bipartitions of k . Let $N \geq 0$ be a sufficiently large integer and consider the multisets corresponding to the above partitions. We also consider the multisets $\hat{Z}_{a,b}^N(\alpha, \beta)$ and $\hat{Z}_{a,b}^N(\alpha', \beta')$, as defined in Lemma 7.9.

Since $M \preceq_L M'$, we have $(\alpha, \beta) \preceq_{a,b} (\alpha', \beta')$ by induction. Recall that this means that $Z_{a,b}^N(\alpha, \beta) \subseteq Z_{a,b}^N(\alpha', \beta')$. By the definition of \preceq , this immediately implies that we also have

$$\hat{Z}_{a,b}^N(\alpha, \beta) \subseteq \hat{Z}_{a,b}^N(\alpha', \beta').$$

Now assume that (I) holds. Then, by Lemmas 7.9 and 7.10, we deduce that

$$Z_{a,b}^N(\lambda, \mu) \subseteq \hat{Z}_{a,b}^N(\alpha, \beta) \subseteq \hat{Z}_{a,b}^N(\alpha', \beta') = Z_{a,b}^N(\lambda', \mu')$$

and, hence, $(\lambda, \mu) \preceq_{a,b} (\lambda', \mu')$, as required.

Now assume that (II) holds. By Lemma 7.1, we have

$$E^{(\lambda, \mu)} \otimes \varepsilon = E^{(\bar{\mu}, \bar{\lambda})} \quad \text{and} \quad E^{(\lambda', \mu')} \otimes \varepsilon = E^{(\bar{\mu}', \bar{\lambda}')}.$$

Arguing as in case (I), we obtain that $(\bar{\mu}', \bar{\lambda}') \preceq_{a,b} (\bar{\mu}, \bar{\lambda})$. But then Proposition 3.12 implies that we also have $(\lambda, \mu) \preceq_{a,b} (\lambda', \mu')$, as required. \square

Remark 7.12. One is tempted to conjecture that the reverse implication in Theorem 7.11 also holds. In the cases where $(a, b) \in \{(1, 1), (1, 0)\}$ or $b > (n - 1)a > 0$, this will be shown in Section 8 below.

However, the reverse implication in Theorem 7.11 does not hold in general. The following example was found by Bonnafé in connection with a somewhat related conjecture in [2, Remark 1.2]. Let $n = 5$, $b = 1$ and $a = 2$. Then one can check that $(\emptyset, 221) \preceq_{2,1} (32, \emptyset)$ but $E^{(\emptyset, 221)}$ and $E^{(32, \emptyset)}$ are not related by \preceq_L .

In any case, the implication in Theorem 7.11 is sufficient to obtain all our applications in Section 9.

Example 7.13. Assume that $a = 0$ and $b > 0$. (See Example 2.7 for the case $a = b = 0$.) Then, by [16, Example 1.3.9], we have

$$\alpha_{E^{(\lambda, \mu)}} = b|\mu| \quad \text{for all bipartitions } (\lambda, \mu) \text{ of } n.$$

Now let (λ, μ) and (λ', μ') be two bipartitions of n . Using similar methods, it is not difficult to show that

$$E^{(\lambda, \mu)} \preceq_L E^{(\lambda', \mu')} \quad \Leftrightarrow \quad |\mu| \geq |\mu'|;$$

furthermore, $E^{(\lambda, \mu)}$ and $E^{(\lambda', \mu')}$ belong to the same family if and only if $|\mu| = |\mu'|$. (As this case is not very interesting for applications, we omit further details; see also [16, Cor. 2.4.12].)

8. EXAMPLES

We keep the notation of the previous section where W_n is a Coxeter group of type B_n and $L: W_n \rightarrow \mathbb{Z}$ is a weight function specified by $b := L(t) \geq 0$ and $a := L(s_i) > 0$ for $1 \leq i \leq n-1$, where $\{t, s_1, \dots, s_{n-1}\}$ are the generators of W_n . In this section, we discuss some examples and further interpretations of the pre-order relation \preceq_L on $\text{Irr}(W_n)$, involving some interesting combinatorics. We begin by considering the “asymptotic case”.

Example 8.1. Assume that $b > (n-1)a > 0$, as in Section 6. Then we claim that:

$$\boxed{E^{(\lambda, \mu)} \preceq_L E^{(\lambda', \mu')} \iff E^{(\lambda, \mu)} \leq_{\mathcal{LR}} E^{(\lambda', \mu')} \iff (\lambda, \mu) \leq (\lambda', \mu')}$$

where \leq is the dominance order on bipartitions; see Example 6.4. In particular, the reverse implication in Remark 2.6 holds. (This is a new result.)

The above equivalences are proved as follows. By Remark 2.6, the first condition implies the second. As already noted in [13, Example 3.7], the second condition implies the third, as a consequence of [15, Prop. 5.4]. So it remains to prove the implication

$$(*) \quad (\lambda, \mu) \leq (\lambda', \mu') \implies E^{(\lambda, \mu)} \preceq_L E^{(\lambda', \mu')}.$$

We begin by noting that, since $b > (n-1)a$, the following simplified version of Lemma 7.10 holds. Let $n = k + l$ where $k \geq 0$, $l \geq 1$. Let (α, β) be a bipartition of k and define $\hat{\alpha}$ to be the partition of $|\alpha| + l$ obtained by increasing the largest l parts of α by 1. Then we have

$$(\dagger) \quad E^{(\alpha, \beta)} \otimes \varepsilon_l \rightsquigarrow_L E^{(\hat{\alpha}, \beta)}.$$

This has already been noted in [8, Prop. 5.2]. Indeed, by Lemma 7.6, we certainly have $E^{(\alpha, \beta)} \otimes \varepsilon_l \uparrow E^{(\hat{\alpha}, \beta)}$. It remains to use a simplified formula for the \mathbf{a} -function, which can now be written as follows (see [15, Example 3.6]):

$$\mathbf{a}_{E^{(\lambda, \mu)}} = (n(\lambda) + 2n(\mu) - n(\bar{\mu}))a + |\mu|b.$$

After these preparations, we can now turn to the proof of $(*)$. It will certainly be sufficient to assume that (λ, μ) is adjacent to (λ', μ') in the dominance order. Also note that, if $(\lambda, \mu) \leq (\lambda', \mu')$, then $|\lambda| \leq |\lambda'|$. According to Remark 6.6, this leads us to distinguish three cases.

Case 1. We have $\mu = \mu'$ and λ is obtained from λ' by decreasing one part by 1 and increasing one part by 1. More precisely, there are indices $1 \leq l < j \leq N$ such that, if we write $\lambda' = (\lambda'_1 \geq \lambda'_2 \geq \dots \geq \lambda'_N)$, then

$$\lambda = (\lambda'_1 \geq \dots \geq \lambda'_{l-1} \geq \lambda'_l - 1 \geq \lambda'_{l+1} \geq \dots \geq \lambda'_{j-1} \geq \lambda'_j + 1 \geq \lambda'_{j+1} \geq \dots \geq \lambda'_N).$$

Let ν be the partition obtained by decreasing the first l parts of λ' by 1. Then notice that λ can be obtained by increasing l parts of ν by 1. Now consider the representation $E^{(\nu, \mu)}$ of W_k where $k = n - l$. By Lemma 7.6 and (\dagger) , we have

$$E^{(\nu, \mu)} \otimes \varepsilon_l \uparrow E^{(\lambda, \mu)} \quad \text{and} \quad E^{(\nu, \mu)} \otimes \varepsilon_l \rightsquigarrow_L E^{(\lambda', \mu)}.$$

This means that $E^{(\lambda, \mu)} \preceq_L E^{(\lambda', \mu)}$, as required.

Case 2. We have $\lambda = \lambda'$ and μ is obtained from μ' by increasing one part by 1 and decreasing one part by 1. Then the bipartitions $(\bar{\mu}', \bar{\lambda}')$ and $(\bar{\mu}, \bar{\lambda})$ are related as in Case 1. So we can conclude that

$$E^{(\lambda', \mu')} \otimes \varepsilon = E^{(\bar{\mu}', \bar{\lambda}')} \preceq_L E^{(\bar{\mu}, \bar{\lambda})} = E^{(\lambda, \mu)} \otimes \varepsilon,$$

where we also used Lemma 7.1. By the definition of \preceq_L , it is then clear that $E^{(\lambda, \mu)} \preceq_L E^{(\lambda', \mu')}$ as required.

Case 3. We have $|\lambda| < |\lambda'|$ and (λ, μ) is obtained from (λ', μ') by decreasing one part of λ' by 1 and increasing one part of μ' by 1. The same statement then also holds, of course, for $(\bar{\mu}, \bar{\lambda})$ and $(\bar{\mu}', \bar{\lambda}')$. In particular, $\bar{\mu}$ is not the empty partition. Let l be the number of (non-zero) parts of $\bar{\mu}$ and let ν be the partition obtained by decreasing all (non-zero) parts of $\bar{\mu}$ by 1. Then notice that $\bar{\mu}'$ can be obtained by increasing some parts of ν by 1. Now consider the representation $E^{(\nu, \bar{\lambda})}$ of W_k where $k = n - l$. By Lemma 7.6 and (\dagger) , we have

$$E^{(\nu, \bar{\lambda})} \otimes \varepsilon_l \uparrow E^{(\bar{\mu}', \bar{\lambda}')} \quad \text{and} \quad E^{(\nu, \bar{\lambda})} \otimes \varepsilon_l \rightsquigarrow_L E^{(\bar{\mu}, \bar{\lambda})}.$$

Using Lemma 7.1, we see that $E^{(\lambda, \mu)} \preceq_L E^{(\lambda', \mu')}$ as required.

Example 8.2. Assume that $a = 1$ and $b \in \{0, 1\}$, as in Example 3.7. We claim that then the reverse implication in Theorem 7.11 also holds, that is, we have:

$$\boxed{E^{(\lambda, \mu)} \preceq_L E^{(\lambda', \mu')} \Leftrightarrow (\lambda, \mu) \preceq_{1,b} (\lambda', \mu')}.$$

Indeed, by the discussion in Example 3.7, it will be sufficient to prove the reverse implication in Theorem 7.11 assuming that (λ, μ) and (λ', μ') are “ $(1, b)$ -special”. But this has already been done by Spaltenstein [23, §4], using an explicit construction which is similar to, but much more ingenious than the one in Example 8.1.

Example 8.3. Let $a = 2$ and $b \geq 1$ be odd. Then $b = 2r + b'$ where $r \geq 0$ and $b' = 1$. This choice of parameters naturally arises from the representation theory of the finite unitary groups $G = \text{GU}_m(\mathbb{F}_q)$ where $m = 2n + \frac{1}{2}r(r+1)$; see [2, Remark 1.1] and the references there.

Let (λ, μ) be a bipartition of n . The corresponding multiset $Z_{2,b}(\lambda, \mu)$ is formed by the $2N + r$ entries

$$\begin{aligned} 2(\lambda_i + N + r - i) + 1 & \quad (1 \leq i \leq N + r), \\ 2(\mu_i + N - i) & \quad (1 \leq i \leq N). \end{aligned}$$

Since these entries are all distinct, we can regard this multiset as the set of β -numbers of a partition, which we denote by $\pi_b(\lambda, \mu)$. Setting $a = 2$ and $b' = 1$ in the right-hand side of the formula in (M1), we obtain

$$na + N^2a + N(b - a) + \binom{r}{2}a + rb' = 2n + \frac{1}{2}r(r+1) + \binom{2N+r}{2}.$$

This means that $\pi_b(\lambda, \mu)$ is a partition of $2n + \frac{1}{2}r(r+1)$. Further note that two partitions are related by the dominance order if and only if the corresponding sets of β -numbers (arranged in decreasing order) are related by the dominance order. Hence, in this case, we can restate the implication in Theorem 7.11 as follows:

$$\boxed{E^{(\lambda, \mu)} \preceq_L E^{(\lambda', \mu')} \Rightarrow \pi_b(\lambda, \mu) \trianglelefteq \pi_b(\lambda', \mu')}.$$

Although the setting is somewhat different, a statement of this kind has been conjectured by Bonnafé et al. [2, Remark 1.2].

Example 8.4. Assume that $a = b = 1$, that is, we are in the “equal parameter case”. Then W_n is the Weyl group of the algebraic group $G = \text{SO}_{2n+1}(F)$, where we can take the field F to be \mathbb{C} or $\overline{\mathbb{F}}_p$ for a prime $p > 2$. In this setting, by the main results of [13], the pre-order relation \preceq_L admits a geometric interpretation in terms

of the Zariski closure relation among the unipotent classes of G . This can also be used to obtain a combinatorial description of \preceq_L which, however, looks different from that in Theorem 7.11 ! Let us briefly explain what is happening here. The Springer correspondence yields a map

$$\mathrm{Irr}(W_n) \rightarrow \{\text{set of unipotent classes of } G\}, \quad E \mapsto O_E;$$

see [13, §5] and the references there. Now assume that $E, E' \in \mathrm{Irr}(W_n)$ are “special”, that is, they are labelled by $(1, 1)$ -special bipartitions in the sense of Example 3.7. Then, by [13, Theorem 4.10 and Corollary 5.5], we have:

$$E \preceq_L E' \quad \Leftrightarrow \quad E \leq_{\mathcal{LR}} E' \quad \Leftrightarrow \quad O_E \subseteq \overline{O_{E'}},$$

where the bar denotes Zariski closure. Now, the Springer correspondence is described explicitly as follows. Consider the map

$$\{\text{bipartitions of } n\} \rightarrow \{\text{partitions of } 2n+1\}, \quad (\lambda, \mu) \mapsto \pi_3(\lambda, \mu),$$

defined in terms of multisets $Z_{2,3}^N(\lambda, \mu)$ as in Example 8.3. (Note that these are not the multisets associated with $a = b = 1$.) Since $G \subseteq \mathrm{GL}_{2n+1}(F)$, every unipotent class of G consists of matrices of a fixed Jordan type which is specified by a partition of $2n+1$. It is known that this partition determines the unipotent class in G ; see [5, §13.1]. Then, as explained in [5, §13.3], we have

$$O_{E(\lambda, \mu)} = \text{unipotent class consisting of matrices of Jordan type } \pi_3(\lambda, \mu).$$

Furthermore, it is known that the closure relation among the unipotent classes of G is given by the dominance order among the partitions labelling the unipotent classes; see [5, §13.4]. Hence, we conclude:

$$\boxed{E^{(\lambda, \mu)} \preceq_L E^{(\lambda', \mu')} \quad \Leftrightarrow \quad \pi_3(\lambda, \mu) \leq \pi_3(\lambda', \mu')}$$

where $(\lambda, \mu), (\lambda', \mu')$ are $(1, 1)$ -special. Thus, via the Springer correspondence, we have obtained a new combinatorial description of \preceq_L . Consequently, by Example 8.2, the following equivalence must be true for $(1, 1)$ -special bipartitions (λ, μ) and (λ', μ') :

$$\pi_3(\lambda, \mu) \leq \pi_3(\lambda', \mu') \quad \Leftrightarrow \quad (\lambda, \mu) \preceq_{1,1} (\lambda', \mu').$$

This can, of course, also be checked directly; see Example 8.5 below.

Finally, W_n can also be regarded as the Weyl group of $G = \mathrm{Sp}_{2n}(F)$. In this case, the Springer correspondence is described using the map

$$\{\text{bipartitions of } n\} \rightarrow \{\text{partitions of } 2n\}, \quad (\lambda, \mu) \mapsto \pi_1(\lambda, \mu),$$

defined in terms of multisets $Z_{2,1}^N(\lambda, \mu)$. Then, arguing as above, one finds that

$$\boxed{E^{(\lambda, \mu)} \preceq_L E^{(\lambda', \mu')} \quad \Leftrightarrow \quad \pi_1(\lambda, \mu) \leq \pi_1(\lambda', \mu')}$$

where $(\lambda, \mu), (\lambda', \mu')$ are $(1, 1)$ -special. Again, for such bipartitions, it must be true that $\pi_1(\lambda, \mu) \leq \pi_1(\lambda', \mu') \Leftrightarrow (\lambda, \mu) \preceq_{1,1} (\lambda', \mu')$.

Example 8.5. Here we give a direct combinatorial proof for the following equivalence that we encountered in Example 8.4 above:

$$Z_{1,1}^N(\lambda, \mu) \leq Z_{1,1}^N(\lambda', \mu') \quad \Leftrightarrow \quad Z_{2,3}^N(\lambda, \mu) \leq Z_{2,3}^N(\lambda', \mu')$$

where (λ, μ) and (λ', μ') are $(1, 1)$ -special bipartitions of n . This will also be a good illustration of how to deal with the multisets $Z_{a,b}^N(\lambda, \mu)$. First, we need some preparations. Let (λ, μ) be $(1, 1)$ -special and write

$$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{N+1} \geq 0), \quad \mu = (\mu_1 \geq \mu_2 \geq \dots \geq \mu_N \geq 0)$$

for some $N \geq 0$. Then $Z_{1,1}^N(\lambda, \mu) = \{z_1, z_2, \dots, z_{2N+1}\}$ where

$$\begin{aligned} z_{2i-1} &= \lambda_i + N + 1 - i & (1 \leq i \leq N+1), \\ z_{2i} &= \mu_i + N - i & (1 \leq i \leq N). \end{aligned}$$

Since (λ, μ) is $(1, 1)$ -special, we have $z_1 \geq z_2 \geq \dots \geq z_{2N+1} \geq 0$. Then note that

$$Z_{2,3}^N(\lambda, \mu) = \{\tilde{z}_1, \tilde{z}_2, \dots, \tilde{z}_{2N+1}\}$$

where the sequence $(\tilde{z}_i)_{1 \leq i \leq 2N+1}$ is defined as follows:

$$\tilde{z}_i = \begin{cases} 2z_i + 1 & \text{if } i \text{ is odd and } z_{i-1} > z_i, \\ 2z_i & \text{if } i \text{ is even and } z_i > z_{i+1}, \\ 2z_i + 1 & \text{if } i \text{ is even and } z_i = z_{i+1}, \\ 2z_i & \text{if } i \text{ is odd and } z_i = z_{i-1}. \end{cases}$$

Since $z_i \geq z_{i+1}$ for all i , one immediately checks that $\tilde{z}_i \geq \tilde{z}_{i+1}$ for all i . For $d \in \{1, \dots, 2N+1\}$, define

$$\varepsilon_d = \begin{cases} 1 & \text{if } d \text{ is even and } z_d = z_{d+1}, \\ 0 & \text{otherwise.} \end{cases}$$

Then we find that

$$(*) \quad \sum_{1 \leq i \leq d} \tilde{z}_i = \varepsilon_d + \lfloor (d+1)/2 \rfloor + 2 \left(\sum_{1 \leq i \leq d} z_i \right),$$

where $\lfloor x \rfloor$ denotes the integer part of x . Now let (λ', μ') also be $(1, 1)$ -special and define

$$Z_{1,1}^N(\lambda', \mu') = \{z'_1, z'_2, \dots, z'_{2N+1}\} \quad \text{and} \quad Z_{2,3}^N(\lambda', \mu') = \{\tilde{z}'_1, \tilde{z}'_2, \dots, \tilde{z}'_{2N+1}\}$$

analogously. If $Z_{2,3}^N(\lambda, \mu) \leq Z_{2,3}^N(\lambda', \mu')$, then

$$\sum_{1 \leq i \leq d} \tilde{z}_i \leq \sum_{1 \leq i \leq d} \tilde{z}'_i \quad \text{for } 1 \leq d \leq 2N+1.$$

Using $(*)$, we deduce that $\varepsilon_d + 2 \sum_{1 \leq i \leq d} z_i \leq \varepsilon'_d + 2 \sum_{1 \leq i \leq d} z'_i$ for all d . This certainly implies that

$$\sum_{1 \leq i \leq d} z_i \leq \sum_{1 \leq i \leq d} z'_i \quad (\text{for all } d) \quad \text{and so} \quad Z_{1,1}^N(\lambda, \mu) \leq Z_{1,1}^N(\lambda', \mu').$$

Conversely, assume that $Z_{1,1}^N(\lambda) \leq Z_{1,1}^N(\lambda', \mu')$. Let $d \geq 1$. Then $(*)$ shows:

$$\varepsilon_d \leq \varepsilon'_d \quad \text{or} \quad \sum_{1 \leq i \leq d} z_i < \sum_{1 \leq i \leq d} z'_i \quad \Rightarrow \quad \sum_{1 \leq i \leq d} \tilde{z}_i \leq \sum_{1 \leq i \leq d} \tilde{z}'_i.$$

So the only case which requires an extra argument is when

$$\varepsilon_d = 1, \quad \varepsilon'_d = 0 \quad \text{and} \quad \sum_{1 \leq i \leq d} z_i = \sum_{1 \leq i \leq d} z'_i.$$

Since $\varepsilon_d = 1$, we have $z_d = z_{d+1}$ and d is even. We have

$$\sum_{1 \leq i \leq d+1} z_i \leq \sum_{1 \leq i \leq d+1} z'_i \quad \text{and so} \quad z_d = z_{d+1} \leq z'_{d+1} < z'_d,$$

where the latter inequality holds since $\varepsilon'_d = 0$. On the other hand, we also have

$$\sum_{1 \leq i \leq d-1} z_i \leq \sum_{1 \leq i \leq d-1} z'_i \quad \text{and so} \quad z_d \geq z'_d,$$

a contradiction. Hence, the special case does not occur. This completes the proof of the desired equivalence.

Example 8.6. Assume that $a = 1$ and $b = 0$. We shall now explain how this case is related to groups of type D_n . Let $\tilde{W}_n \subseteq W_n$ be the subgroup generated by the reflections $\{ts_1t, s_1, \dots, s_{n-1}\}$ and let \tilde{L} denote the restriction of L to \tilde{W}_n . Then \tilde{W}_n is of type D_n and \tilde{L} is the usual length function on \tilde{W}_n ; see, for example, [17, §1.4]. The irreducible representations of \tilde{W}_n are classified as follows. Given a bipartition (λ, μ) of n , we denote by $E^{[\lambda, \mu]}$ the restriction of $E^{(\lambda, \mu)} \in \text{Irr}(W_n)$ to \tilde{W}_n . Then we have (see [17, 5.6.1]):

- If $\lambda \neq \mu$, then $E^{[\lambda, \mu]} = E^{[\mu, \lambda]}$ is an irreducible representation of $\text{Irr}(\tilde{W}_n)$.
- If $\lambda = \mu$, then $E^{[\lambda, \lambda]} = E^{[\lambda, +]} \oplus E^{[\lambda, -]}$ where $E^{[\lambda, +]}$ and $E^{[\lambda, -]}$ are non-isomorphic irreducible representations of \tilde{W}_n . (This can only occur if n is even.)

Furthermore, all irreducible representations of \tilde{W}_n arise in this way. We shall say that $\tilde{E} \in \text{Irr}(\tilde{W}_n)$ is “special” if \tilde{E} is a constituent of $E^{[\lambda, \mu]}$ where (λ, μ) is $(1, 0)$ -special in the sense of Example 3.7. This coincides with Lusztig’s definition [20, Chap. 4]. In particular, each family of $\text{Irr}(\tilde{W}_n)$ contains a unique special representation and so it is enough to describe $\preceq_{\tilde{L}}$ for special representations.

Now \tilde{W}_n is the Weyl group of the algebraic group $\tilde{G} = \text{SO}_{2n}(F)$ where, as above, F is \mathbb{C} or \mathbb{F}_p for a prime $p > 2$. Again, the Springer correspondence yields a map

$$\text{Irr}(\tilde{W}_n) \rightarrow \{\text{set of unipotent classes of } \tilde{G}\}, \quad \tilde{E} \mapsto \tilde{O}_{\tilde{E}},$$

which is explicitly described in [5, §13.3]. By the main results of [13], we have

$$\tilde{E} \preceq_{\tilde{L}} \tilde{E}' \quad \Leftrightarrow \quad \tilde{E} \leq_{\mathcal{LR}} \tilde{E}' \quad \Leftrightarrow \quad \tilde{O}_{\tilde{E}} \subseteq \overline{\tilde{O}_{\tilde{E}'}}$$

for any $\tilde{E}, \tilde{E}' \in \text{Irr}(\tilde{W}_n)$ which are special. By Spaltenstein [23, §4], the condition on the right hand side can be expressed using $\preceq_{1,0}$. More precisely, let (λ, μ) and (λ', μ') be $(1, 0)$ -special bipartitions of n such that \tilde{E} is a constituent of $E^{[\lambda, \mu]}$ and \tilde{E}' is a constituent of $E^{[\lambda', \mu']}$. Here, (λ, μ) and (λ', μ') are uniquely determined. (Just note that, if both (λ, μ) and (μ, λ) are $(1, 0)$ -special, then $\lambda = \mu$.) Then

$$\tilde{E} \preceq_{\tilde{L}} \tilde{E}' \quad \Leftrightarrow \quad \begin{cases} \tilde{E} = \tilde{E}' & \text{if } \lambda = \lambda' = \mu = \mu', \\ (\lambda, \mu) \preceq_{1,0} (\lambda', \mu') & \text{otherwise.} \end{cases}$$

Thus, $\preceq_{\tilde{L}}$ is essentially determined by $\preceq_{1,0}$, where some special care is required when comparing the two representations $E^{[\lambda, \pm]}$ with each other.

9. CONCLUDING REMARKS

We return to the general setting where W is any finite Coxeter group and $L: W \rightarrow \mathbb{Z}$ is a weight function such that $L(s) \geq 0$ for $s \in S$. Having established the results in Section 7, we can now formulate some properties of the pre-order relation \preceq_L which hold in complete generality.

Theorem 9.1. *Let $E, E' \in \text{Irr}(W)$. If $E \preceq_L E'$, then $\mathbf{a}_{E'} \leq \mathbf{a}_E$, with equality only if E, E' belong to the same family.*

Proof. Using Remark 2.8, it is sufficient to prove this in the case where (W, S) is irreducible. So let us assume that (W, S) is irreducible. If $L(s) = 0$ for all $s \in S$, then the assertions trivially hold by Example 2.7. Assume now that $L(s) > 0$ for some $s \in S$.

If W is of type $A_n, D_n, H_3, H_4, E_6, E_7, E_8$ or $I_2(m)$ (m odd), then we are in the equal parameter case and the assertions hold by [13, Example 3.5 and Prop. 4.4].

If W is of type $I_2(m)$ (m even) or F_4 , the assertions can be checked by explicit computations; see [9] and [13, Examples 3.5 and 3.6]. If $L(s) = 0$ for some $s \in S$, see [16, Example 2.2.8 and Remark 2.4.13] where similar verifications have been performed with respect to the pre-order relation $\leq_{\mathcal{LR}}$.

Finally, assume that W is of type B_n with parameters a, b as in Section 7. If $a > 0$, the assertions follow from Theorem 7.11 and Remark 3.9, in combination with Propositions 7.2 and 7.4. For the case $a = 0$, see Example 7.13. \square

Corollary 9.2. *Let $E, E' \in \text{Irr}(W)$. Then E and E' belong to the same family if and only if $E \preceq_L E'$ and $E' \preceq_L E$.*

Proof. The “only if” part is clear by the definitions, as already mentioned in [13, Remark 2.11]. The “if” part now immediately follows from Theorem 9.1. \square

Corollary 9.3. *Let $E, E' \in \text{Irr}(W)$. If $E \preceq_L E'$, then $\omega_L(E) \leq \omega_L(E')$, with equality only if E, E' belong to the same family.*

Proof. Assume that $E \preceq_L E'$. By the definition of \preceq_L , it is clear that then we also have $E' \otimes \varepsilon \preceq_L E \otimes \varepsilon$. Hence, using Remark 2.2 and Theorem 9.1, we deduce that $\omega_L(E) \leq \omega_L(E')$, with equality only if E, E' belong to the same family. \square

Remark 9.4. Assume that W is a Weyl group and we are in the equal parameter case. For $E \in \text{Irr}(W)$, let $D_E(u) \in \mathbb{Q}[u]$ (where u is an indeterminate) be the corresponding generic degree, defined in terms of the associated Iwahori–Hecke algebra; see, for example, [17, Cor. 9.3.6]. Then the invariant \mathbf{a}_E is given by

$$D_E(u) = f_E^{-1} u^{\mathbf{a}_E} + \text{linear combination of higher powers of } u,$$

where f_E is a non-zero integer; see Lusztig [20, 4.1]. Furthermore, we have

$$D_E(u) = f_E^{-1} u^{\mathbf{A}_E} + \text{linear combination of lower powers of } u,$$

for some integer $\mathbf{A}_E \geq 0$. By [17, Prop. 9.4.3], we have $\omega_L(E) = L(w_0) - (\mathbf{a}_E + \mathbf{A}_E)$. Hence, Corollary 9.3 implies that

$$E \preceq_L E' \quad \Rightarrow \quad \mathbf{a}_{E'} + \mathbf{A}_{E'} \geq \mathbf{a}_E + \mathbf{A}_E,$$

with equality only if E, E' belong to the same family. An analogous statement, where “families” and “ \preceq_L ” are replaced by “two-sided cells” and the relation “ $\leq_{\mathcal{LR}}$ ” (as referred to in Remark 2.6), plays a role in Ginzburg et al., [18, Prop. 6.7].

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